PSL UNIVERSITÉ PARIS

THÈSE DE DOCTORAT
DE L'UNIVERSITÉ PSL

Préparée à l'École Normale Supérieure

Diagonal restriction and denominators of some Eisenstein cohomology classes

Soutenue par

Romain BRANCHEREAU

Le 25 octobre 2022

École doctorale n°386

Sciences Mathématiques de Paris Centre

Spécialité

Mathématiques

Composition du jury :

Claudia ALFES-NEUMANN Professor, Bielefeld University	*Examinatrice*
Nicolas BERGERON Professeur, École Normale Supérieure	*Directeur de thèse*
Henri DARMON Professor, McGill	*Rapporteur*
Luis GARCIA Lecturer, University College London	*Examinateur*
Özlem IMAMOGLU Professor, ETH Zürich	*Examinatrice*
Yingkun LI Assistant Professor, TU Darmstadt	*Rapporteur*
Ariane MÉZARD Professeur, École Normale Supérieure	*Présidente du jury*

ENS | PSL

Remerciements

Je souhaite tout d'abord remercier chaleureusement mes directeurs de thèse Nicolas et Luis, sans eux rien n'aurait été possible. Ils m'ont introduit à un sujet de thèse passionant qui mélange arithmétique et géométrie des espaces localement symétriques. Je les remercie infiniment pour leur disponibilité sans faille et leur bienveillance lors des innombrables discussions que nous avons eu ces trois dernières années et au cours desquelles j'ai tant appris.

Je suis très reconnaissant envers Henri Darmon et Yingkun Li d'avoir accepté d'être rapporteurs, ainsi qu'envers Claudia Alfes-Neumann, Özlem Imamoglu et Ariane Mézard d'avoir accepté d'être examinatrices[1].

J'ai profité d'excellentes conditions au DMA pour rédiger cette thèse. Grâce à la bonne humeur de tout le monde ce fût toujours un grand bonheur de venir travailler au DMA! Je remercie tout particulièrement: Antoine, Ariane, Cyril, Fabienne, Farrel, François, Isabelle, Gaetan, les Nicolas, les Oliviers, Paul, Zaïna, Zoé et tous les doctorants et postdoc.

Je remercie Henri Darmon et Jan Vonk pour des discussions qui furent très utiles - voire décisives - autour de cette thèse, et Javier sans qui je n'aurais certainement jamais fait ma thèse auprès de Nicolas. Je remercie aussi la FMSP et tout particulièrement Ariela Briani pour la mise en place du programme Cofund qui a financé[2] ma thèse.

Parmis les nombreux doctorants et postdoc du DMA je suis ravi d'avoir fait la connaissance d'Anthony, Arnaud V et Arnaud E, Béranger, Coline, Eduardo, Léonard, Nima, Olivier, Pierre, Samuel, Stefan, Vadim et Yusuke. J'ai pu compter sur eux tous pour boire des coups à la fin d'une journée de dur labeur (ou pas), et j'espère bien le refaire très vite! Mention spéciale à mon ami Olivier avec qui j'ai eu le plaisir de partager tant de chouettes moments à Paris depuis notre premier jour de thèse jusqu'à notre soutenance, en plus d'avoir partagé le plus beau bureau du DMA.

Ces années de thèse ne fûrent pas seulement studieuses mais aussi sportives, et ce ne sont pas

[1] I am very grateful to Henri Darmon and Yingkun Li for accepting to be referees for my thesis, and to Claudia Alfes-Neumann, Özlem Imamoglu and Ariane Mézard for accepting to be part of the jury.

[2] I was funded from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement N$^{\underline{o}}$754362 .

les Bonnets Rouges du PSC qui diront le contraire! Merci à eux pour les supers moments passés dans et hors des bassins, en particulier Claire, Elsa, François, JC, Hugo, Karim, les Mousquetaires Paul et Lilian, Morgane, Nathalie, Valérie, Vincent.

Merci à mes amis dont les bons moments partagés ont égayé ces dernières années: Adrien, Jules, Claudia, Jonas, Quentin, Roxane, ainsi qu'Alex, Haowen et Matthieu entre autres. Enfin merci à ma famille et tout particulièrement mes parents, pour leur présence et leurs encouragements.

Pour finir je remercie toute l'équipe du film *Le théorème de Marguerite*, ainsi qu'Ariane de nous avoir permis d'y participer avec Anthony. On s'est bien amusé! En plus d'avoir eu la chance de faire l'expérience géniale d'un plateau de tournage, grâce à eux le spectateur attentif[3] pourra apercevoir le résultat principal de cette thèse sur grand écran au cinéma!

[3]Si je n'ai pas été coupé au montage, on devrait me voir dans une scène de figuration au début du film. Alors que Marguerite marche dans le couloir, on m'y voit au tableau dans l'espace Cartan en train d'écrire le Théorème B de l'introduction.

Contents

Chapter 1

Introduction

The common theme of this thesis is the study of the cohomology of some locally symmetric spaces in relation to theta and Eisenstein series. In a first part (Chapters 3 and 4) we consider a locally symmetric space M_K associated to an orthogonal group and Kudla-Millson theta series. In a second part (Chapter 5) we consider the locally symmetric space associated to $\mathrm{SL}_2(K)$ for an imaginary quadratic field K and Harder's Eisenstein cohomology. In both case the cohomology classes that we work with can be constructed using a certain differential form due to Mathai and Quillen.

Part 1

Locally symmetric spaces associated to orthogonal groups. Before stating the results let us explain the general setting. Let $(X_{\mathbb{Q}}, Q)$ be a rational quadratic space of signature (p,q) and let $H \coloneqq \mathrm{SO}(Q)$ be its orthogonal group. Let

$$\mathbb{D} \coloneqq \left\{ z \subset X_{\mathbb{R}} \quad \text{oriented} \quad \middle| \quad \dim(z) = q, \quad Q\big|_z < 0 \right\}$$

be the space of *oriented* negative q-planes in $X_{\mathbb{R}} = X_{\mathbb{Q}} \otimes \mathbb{R}$. It is a non-connected symmetric space of dimension pq, and we denote by $\mathbb{D}^+$ one of its two connected components. In low dimensions the space $\mathbb{D}^+$ can for example be identified with $\mathbb{R}_{>0}$ when $(p,q) = (1,1)$, with $\mathbb{H}$ when $(p,q) = (2,1)$ or with $\mathbb{H} \times \mathbb{H}$ when $(p,q) = (2,2)$. We will consider the latter example in Section 4.4.

Let $H(\mathbb{R})^+$ be the connected component of the identity of $H(\mathbb{R})$. It acts transitively on $\mathbb{D}^+$, so that we can identify $H(\mathbb{R})^+/K_\infty$ with $\mathbb{D}^+$, where K_∞ is a maximal compact connected subgroup of $H(\mathbb{R})^+$ that is isomorphic to $\mathrm{SO}(p) \times \mathrm{SO}(q)$. Let $\varphi_f \in \mathscr{S}(X_{\mathbb{A}_f})$ be a finite Schwartz function on $X_{\mathbb{A}_f} = X_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{A}_f$, which is K_f-invariant for some open compact $K_f \subset H(\mathbb{A}_f)$ *i.e.*

$$\varphi_f(k\mathbf{x}) = \varphi_f(\mathbf{x}) \qquad \forall k \in K_f, \qquad \forall \mathbf{x} \in X_{\mathbb{A}_f}.$$

The space we want to study is the double coset space

$$M_K := H(\mathbb{Q})\backslash H(\mathbb{A})/K \tag{1.0.1}$$

where $K := K_\infty K_f$. It is known [PR94, Theorem. 5.1] that the double coset $H(\mathbb{Q})^+\backslash H(\mathbb{A}_f)/K_f$ is finite, so that we can find representatives $h_1, \dots, h_r \in H(\mathbb{A}_f)$ such that

$$H(\mathbb{A}_f) = \bigsqcup_{i=1}^{r} H(\mathbb{Q})^+ h_i K_f. \tag{1.0.2}$$

The space M_K is a disjoint union

$$M_K = \bigsqcup_{i=1}^{r} M_{h_i}$$

of locally symmetric spaces $M_h := \Gamma_h\backslash\mathbb{D}^+$ of dimension pq, where Γ_h is the image in $H^{\mathrm{ad}}(\mathbb{Q}) = H(\mathbb{Q})/Z(\mathbb{Q})$ of

$$\Gamma'_h := H(\mathbb{Q})^+ \cap hK_f h^{-1}.$$

Note that $\Gamma'_h \subset H(\mathbb{Q})^+$ preserves the set $\mathcal{L}_h(\varphi_f)$ in $X_\mathbb{Q}$ defined by

$$\mathcal{L}_h(\varphi_f) := \left\{ \mathbf{x} \in X_\mathbb{Q} | \, \varphi_f(h^{-1}\mathbf{x}) \neq 0 \right\}.$$

If φ_f is the characteristic function of an adelic lattice, then $\mathcal{L}_h(\varphi_f)$ is a lattice in $X_\mathbb{Q}$

In general M_K is non-compact and we let $\overline{M_K}$ be a compactification. To simplify the notation, suppose for the rest of the introduction that $r = 1$ and let $\Gamma = \Gamma_h$ for $h = 1$. In that case, the subgroup $\Gamma \subset H(\mathbb{Q})^+$ preserves $\mathcal{L}(\varphi_f) = X_\mathbb{Q} \cap \mathrm{supp}(\varphi_f)$ and $M_K = \Gamma\backslash\mathbb{D}^+$ is connected.

Remark 1.0.1. The decomposition (1.0.2) holds for an arbitrary algebraic group H. When $K_f = H(\widehat{\mathbb{Z}})$, then the integer $\mathrm{cl}(H) := r$ is called the class number of H. When H is the orthogonal group of a quadratic form Q defined over $\mathbb{Z}$, one can show [PR94, p. 450] that the class number of H equals the number of classes in the genus of Q. In particular we have $r > 1$ in general. However the assumption $r = 1$ is still reasonable. For example in the special case that we will consider in Subsection 4.4 we will have $r = 1$. This is due to a special isomorphism between H and $\mathrm{SL}_2 \times \mathrm{SL}_2$, and to the strong approximation of SL_2.

Special cycles. For every positive vector $\mathbf{x} \in X_\mathbb{Q}$ there is a totally geodesic submanifold $\mathbb{D}^+_\mathbf{x} \subset \mathbb{D}^+$ of codimension q, consisting of the negative q-planes that are orthogonal to $\mathbf{x}$. For example if $(p, q) = (2, 1)$ then $\mathbb{D}^+_\mathbf{x}$ is a geodesic in $\mathbb{H}$ and if $(p, q) = (2, 2)$ then $\mathbb{D}^+_\mathbf{x}$ is an embedded upper half plane in $\mathbb{H} \times \mathbb{H}$.

When passing to the quotient M_K, the submanifold $\mathbb{D}_{\mathbf{x}}^+ \subset \mathbb{D}^+$ defines a relative cycle $C_{\mathbf{x}} \in \mathcal{Z}_{pq-q}(\overline{M_K}, \partial\overline{M_K}; \mathbb{R})$ of codimension q and

$$C_n(\varphi_f) := \sum_{\substack{\mathbf{x} \in \Gamma \backslash X_{\mathbb{Q}} \\ Q(\mathbf{x},\mathbf{x})=2n}} \varphi_f(\mathbf{x}) C_{\mathbf{x}} \in \mathcal{Z}_{pq-q}(\overline{M_K}, \partial\overline{M_K}; \mathbb{R})$$

for $n \in \mathbb{Q}_{>0}$. We call them *special cycles*.

The Kudla-Millson form. Let $\mathscr{S}(X_{\mathbb{R}})$ be the space of Schwartz function on $X_{\mathbb{R}}$. The group $H(\mathbb{R})^+$ acts on $\mathscr{S}(X_{\mathbb{R}})$ by $(hf)(\mathbf{x}) := f(h^{-1}\mathbf{x})$. Let $\Gamma_{\mathbf{x}}$ denote the stabilizer of $\mathbf{x}$ in Γ. We can view $\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+$ as a rank q vector bundle over $\Gamma_{\mathbf{x}} \backslash \mathbb{D}_{\mathbf{x}}^+$, so that the natural embedding $\Gamma_{\mathbf{x}} \backslash \mathbb{D}_{\mathbf{x}}^+ \subset \Gamma_{\mathbf{x}} \backslash \mathbb{D}^+$ is the zero section.

In [KM86], Kudla and Millson construct a closed $H(\mathbb{R})^+$-invariant differential form

$$\varphi_{KM} \in \left[\Omega^q(\mathbb{D}^+) \otimes \mathscr{S}(X_{\mathbb{R}})\right]^{H(\mathbb{R})^+},$$

where $H(\mathbb{R})^+$ acts on the Schwartz space $\mathscr{S}(X_{\mathbb{R}})$ as above and on $\Omega^q(\mathbb{D}^+) \otimes \mathscr{S}(X_{\mathbb{R}})$ by

$$h \cdot (\omega \otimes f) := h^* \omega \otimes (h^{-1} f).$$

In particular $\varphi_{KM}(\mathbf{x})$ is a $\Gamma_{\mathbf{x}}$-invariant form on $\mathbb{D}^+$ and we can view it as a form on $\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+$. The main property of the Kudla-Millson form is its *Thom form property*: if $\omega \in \Omega_c^q(\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+)$ is a compactly supported form, then

$$\int_{\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+} \varphi_{KM}(\mathbf{x}) \wedge \omega = 2^{-\frac{q}{2}} e^{-\pi Q(\mathbf{x},\mathbf{x})} \int_{\Gamma_{\mathbf{x}} \backslash \mathbb{D}_{\mathbf{x}}^+} \omega.$$

Another way to state it is to say that as cohomology classes in $H^q(\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+)$ we have

$$[\varphi_{KM}(\mathbf{x})] = 2^{-\frac{q}{2}} e^{-\pi Q(\mathbf{x},\mathbf{x})} \, \mathrm{PD}(\Gamma_{\mathbf{x}} \backslash \mathbb{D}_{\mathbf{x}}^+), \tag{1.0.3}$$

where $\mathrm{PD}(\Gamma_{\mathbf{x}} \backslash \mathbb{D}_{\mathbf{x}}^+)$ denotes the Poincaré dual of $\Gamma_{\mathbf{x}} \backslash \mathbb{D}_{\mathbf{x}}^+$.

The Kudla-Millson theta lift. In [KM86; KM87; KM90], Kudla and Millson used the form φ_{KM} and the work of Weil [Wei64] on theta series to construct a closed differential form

$$\Theta_{KM}(\tau, \varphi_f) \in \Omega^q(M_K)$$

where $\tau \in \mathbb{H}$. It represents a cohomology class in $H^q(M_K;\mathbb{R})$ and can be paired with a (compact) cycle $C \in \mathcal{Z}_q(M_K;\mathbb{R})$. They showed that the function

$$\tau \longmapsto \int_C \Theta_{KM}(\tau,\varphi_f) \tag{1.0.4}$$

is a holomorphic modular form of weight $\frac{p+q}{2}$. Hence, the Kudla-Millson theta series realizes a lift between the (co)-homology of $\Gamma\backslash\mathbb{D}^+$ and the space of weight $\frac{p+q}{2}$ modular forms. We recall the construction in more detail in Section 2.2. Moreover, we have the Fourier expansion

$$\int_C \Theta_{KM}(\tau,\varphi_f) = \kappa \sum_{n\in\mathbb{Q}_{\geq 0}} \left(\int_C \Theta_n(\varphi_f)\right) e^{2i\pi n\tau},$$

where $\Theta_n(\varphi_f) \in \Omega^q(M_K)$ is a Poincaré dual to $C_n(\varphi_f)$ when $n > 0$ and

$$\kappa \coloneqq \begin{cases} 2 & \text{if } -1 \in K_f \cap H(\mathbb{Q})^+, \\ 1 & \text{otherwise.} \end{cases} \tag{1.0.5}$$

Since C is compact, the Fourier coefficients are equal to the topological intersection numbers $\langle C_n(\varphi_f), C\rangle$ and

$$\int_C \Theta_{KM}(\tau,\varphi_f) = \int_C \Theta_0(\varphi_f) + \kappa \sum_{n\in\mathbb{Q}_{>0}} \langle C_n(\varphi_f), C\rangle e^{2i\pi n\tau}. \tag{1.0.6}$$

The work of Kudla-Millson holds in much greater generality: their construction also works for Siegel modular forms instead of modular forms, and for unitary groups instead of orthogonal groups.

1.1 Kudla-Millson form and the Mathai-Quillen formalism

The goal of Chapter 3 is to show how to recover the Kudla-Millson form from a construction due to Mathai and Quillen [MQ86]. Let E be the tautological bundle over $\mathbb{D}^+$, whose fiber E_z above $z \in \mathbb{D}^+$ is the negative q-plane z itself. We equip this bundle with the metric $-Q\big|_z$ in every fiber, which makes it a metric bundle. It is an $H(\mathbb{R})^+$-equivariant vector bundle of rank q over $\mathbb{D}^+$ and we denote by E_0 the image of the zero section. Since it is $H(\mathbb{R})^+$-equivariant it is in particular $\Gamma_{\mathbf{x}}$-equivariant and we also have a vector bundle $\Gamma_{\mathbf{x}}\backslash E$ over $\Gamma_{\mathbf{x}}\backslash\mathbb{D}^+$. The *Thom class* of the vector bundle is a characteristic class

$$\mathrm{Th}(\Gamma_{\mathbf{x}}\backslash E) \in H^q(\Gamma_{\mathbf{x}}\backslash E, \Gamma_{\mathbf{x}}\backslash(E - E_0))$$

defined by the Thom isomorphism; see Section 2.1.3 for more details. Equivalently, viewing it as a class in $H^q(\Gamma_{\mathbf{x}}\backslash E)$, it is also the Poincaré dual class to $\Gamma_{\mathbf{x}}\backslash E_0$. If $s_{\mathbf{x}}\colon \Gamma_{\mathbf{x}}\backslash \mathbb{D}^+ \longrightarrow \Gamma_{\mathbf{x}}\backslash E$ is a section whose zero locus is $\Gamma_{\mathbf{x}}\backslash \mathbb{D}^+_{\mathbf{x}}$, then

$$s_{\mathbf{x}}^* \operatorname{Th}(\Gamma_{\mathbf{x}}\backslash E) \in H^q\left(\Gamma_{\mathbf{x}}\backslash \mathbb{D}^+, \Gamma_{\mathbf{x}}\backslash(\mathbb{D}^+ - \mathbb{D}^+_{\mathbf{x}})\right).$$

Viewing it as a class in $H^q(\Gamma_{\mathbf{x}}\backslash \mathbb{D}^+)$ it is the Poincaré dual class of $\Gamma_{\mathbf{x}}\backslash \mathbb{D}^+_{\mathbf{x}}$. Since the Poincaré dual class is unique, property (1.0.3) implies that

$$[\varphi_{KM}(\mathbf{x})] = 2^{-\frac{q}{2}} e^{-\pi Q(\mathbf{x},\mathbf{x})} s_{\mathbf{x}}^* \operatorname{Th}(\Gamma_{\mathbf{x}}\backslash E) \in H^q\left(\Gamma_{\mathbf{x}}\backslash \mathbb{D}^+\right),$$

on the level of cohomology.

For arbitrary oriented real Euclidean vector bundles equipped with a Euclidean connection ∇, Mathai and Quillen [MQ86] used the Chern-Weil theory to construct a canonical form representing the Thom class. We call such forms *Thom forms*. In particular we can apply this to the bundle $\Gamma_{\mathbf{x}}\backslash E$ and we denote by $U_{MQ} \in \Omega^q(\Gamma_{\mathbf{x}}\backslash E)$ the canonical Thom form of Mathai and Quillen. After pulling it back by $s_{\mathbf{x}}$ we obtain a form $s_{\mathbf{x}}^* U_{MQ} \in \Omega^q(\Gamma_{\mathbf{x}}\backslash \mathbb{D}^+)$. The main result of Chapter 3 relates this form to the Kudla-Millson form.

Theorem A. *(Theorem 3.2.5) We have* $\varphi_{KM}(\mathbf{x}) = 2^{-\frac{q}{2}} e^{-\pi Q(\mathbf{x},\mathbf{x})} s_{\mathbf{x}}^* U_{MQ}$ *in* $\Omega^q(\Gamma_{\mathbf{x}}\backslash \mathbb{D}^+)$.

For signature $(2,q)$, the spaces are hermitian and the result was obtained by a similar method in [Gar18] using the work of Bismut-Gillet-Soulé.

1.2 Diagonal restriction of Eisenstein series

The starting point of the work presented in Chapter 4 is a result of Darmon-Pozzi-Vonk, that relates the diagonal restriction of an Eisenstein series to intersection numbers of geodesics. Let ψ be a finite order totally odd Hecke character on the narrow class group $\operatorname{Cl}(F)^+$ of a real quadratic field F of discriminant d_F. To such a character one can associate an Eisenstein series $E(\tau_1, \tau_2, \psi)$ which is a Hilbert modular form of parallel weight 1 and level $\mathrm{SL}_2(\mathscr{O})$. Its diagonal restriction $E(\tau, \tau, \psi)$ is a modular form of weight 2 and level $\mathrm{SL}_2(\mathbb{Z})$. As such, it is zero, since this is the only such form. Instead one can look at the p-stabilization $E^{(p)}(\tau_1, \tau_2, \psi)$ for some prime p. The diagonal restriction $E^{(p)}(\tau, \tau, \psi)$ is now a form of weight 2 and level $\Gamma_0(p)$. Moreover, it is non-zero when p is split.

Suppose that p is a split prime and let $Y_0(p)$ be the modular $\Gamma_0(p)\backslash\mathbb{H}$. To a fractional ideal $\mathfrak{a}$ in F and a square root r of d_F modulo p one can associate a closed geodesic $\overline{\mathcal{Q}}_{\mathfrak{a},r}$ in $Y_0(p)$. Let $\overline{\mathcal{Q}}(\psi)$

be the 1-cycle defined by

$$\overline{\mathcal{Q}}(\psi) \coloneqq \sum_{[\mathfrak{a}]\in \mathrm{Cl}(F)^+} \psi(\mathfrak{a})(\overline{\mathcal{Q}}_{\mathfrak{a},r} + \overline{\mathcal{Q}}_{\mathfrak{a},-r}) \in \mathcal{Z}_1(Y_0(p)),$$

and let $\overline{\mathcal{Q}}(0,\infty)$ be the image in $Y_0(p)$ of the geodesic joining 0 to ∞. The following identity is proved in [DPV21, Theorem. A]

$$E^{(p)}(\tau,\tau,\psi) = L^{(p)}(\psi,0) - 2\sum_{n=1}^{\infty} \langle \overline{\mathcal{Q}}(0,\infty), T_n\overline{\mathcal{Q}}(\psi)\rangle e^{2i\pi n\tau} \tag{1.2.1}$$

where $L^{(p)}(\psi,0) = (1-\psi(\mathfrak{p}))(1-\psi(\mathfrak{p}^\sigma))L(\psi,0)$ and T_n is a Hecke operator defined by double cosets. In *loc. cit.* the equality (1.2.1) is proved by computing the intersection numbers and comparing them with the Fourier coefficients of $E^{(p)}(\tau,\tau,\psi)$. Having the construction of Kudla-Millson in mind it is very tempting to ask the following question.

Question. *Can we recover equality* (1.2.1) *by the Kudla-Millson lift?*

The goal of Chapter 4 is to positively answer this question and to generalize the result of Darmon-Pozzi-Vonk to totally real fields. For this we construct a *relative* cycle $C\otimes\psi$ that is a torus on M_K, and such that the integral of the Kudla-Millson theta series over this cycles gives the diagonal restriction of the Eisenstein series.

A space of signature (N,N). At first glance, by comparing Equations (1.0.6) and (1.2.1), one could think that we need to work with the symmetric space $\mathbb{H}$ associated to the orthogonal group of signature $(2,1)$. However the correct setting that we need is a locally symmetric associated to a space of signature $(2,2)$ and exploiting the special isomorphism between $\mathbb{D}^+$ and $\mathbb{H}\times\mathbb{H}$ mentionned earlier. For the generalization a space of signature (N,N), defined as follows.

Let F be a totally real field of degree N with ring of integers $\mathscr{O}$. Let $X_F^0 = F^2$ be the 2-dimensional quadratic F-space with the quadratic form $Q^0(\mathbf{x},\mathbf{y}) = xy' + x'y$ where $\mathbf{x} = (x,x')$ and $\mathbf{y} = (y,y')$ are vectors in F^2. At a place v of $\mathbb{Q}$ let $F_{\mathbb{Q}_v} \coloneqq F_{\mathbb{Q}}\otimes\mathbb{Q}_v$, where we write $F_{\mathbb{Q}}$ instead of F to emphasize that we view F as a $\mathbb{Q}$-algebra. We fix a $\mathbb{Q}$-basis of F that identifies $F_{\mathbb{Q}}$ with $\mathbb{Q}^N$. Let $X_{\mathbb{Q}} \coloneqq \mathrm{Res}_{F/\mathbb{Q}} X_F^0 = F_{\mathbb{Q}}^2$ be the $2N$-dimensional quadratic $\mathbb{Q}$-space with the quadratic form $Q \coloneqq \mathrm{tr}_{F/\mathbb{Q}} Q^0$. The real space $X_{\mathbb{R}}$ is of signature (N,N). Let M_K be the associated double coset space defined in (1.0.1), where K_f is preserving a finite Schwartz function $\varphi_f \in \mathscr{S}(X_{\mathbb{A}_f})$, and consider the Kudla-Millson theta series $\Theta_{KM}(\tau,\varphi_f) \in \Omega^N(M_K)$.

A non-compact cycle $C\otimes\psi$. Let $\psi\colon F^\times\backslash\mathbb{A}_F^\times \longrightarrow U(1)$ be a unitary totally odd Hecke character of finite order. To simplify the notation let us suppose in the introduction that ψ is unramified. Hence the finite part of the Hecke character can be viewed as a character on the narrow class group, as in the beginning of (1.2).

Let $\mathrm{SO}(F^2) \subset \mathrm{GL}_2(F)$ be the orthogonal group of the quadratic space $X_F^0 = F^2$ with the quadratic form defined above. The map

$$\begin{aligned} F^\times &\longrightarrow \mathrm{SO}(F^2) \\ t &\longmapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \end{aligned}$$

yields an isomorphism $\mathrm{SO}(F^2)$ and $F^\times$. On the other hand the group $\mathrm{SO}(F^2)$ can naturally be embedded in $\mathrm{SO}(F^2_{\mathbb{Q}}) \subset \mathrm{GL}_{2N}(\mathbb{Q})$ by restriction of scalars. Composing the two and passing to the adèles gives an embedding

$$h\colon \mathbb{A}_F^\times \hookrightarrow \mathrm{SO}(F^2_{\mathbb{A}}) \subset \mathrm{GL}_{2N}(\mathbb{A}),$$

where $F^2_{\mathbb{A}} = F^2_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{A}$. For K_f large enough we have $h(\widehat{\mathscr{O}}^\times) \subset K_f$ and the embedding h induces an immersion

$$M_{\mathscr{O}} \longrightarrow M_K, \tag{1.2.2}$$

where

$$M_{\mathscr{O}} := F^\times \backslash \mathbb{A}_F^\times / \{\pm 1\}^{N-1} \times \widehat{\mathscr{O}}^\times.$$

The space $M_{\mathscr{O}}$ is not connected. There is a bijection between classes in the narrow class group $\mathrm{Cl}(F)^+$ and connected components of $M_{\mathscr{O}}$. More precisely it can be written as a disjoint union

$$M_{\mathscr{O}} = \bigsqcup_{[\mathfrak{a}] \in \mathrm{Cl}(F)^+} \Gamma \backslash \mathbb{R}^N_{>0}$$

where $\Gamma := \mathscr{O}^{\times,+}$ are the totally positive units in $\mathscr{O}$. The connected components are N-dimensional tori. The image by the immersion (1.2.2) of the connected component corresponding to the class $[\mathfrak{a}] \in \mathrm{Cl}(F)^+$ defines a cycle relative to the boundary

$$C_{\mathfrak{a}} \in \mathcal{Z}_N(\overline{M_K}, \partial \overline{M_K}; \mathbb{R}).$$

We twist it by the Hecke character ψ to obtain a relative cycle

$$C \otimes \psi := \sum_{[\mathfrak{a}] \in \mathrm{Cl}(F)^+} \psi(\mathfrak{a}) C_{\mathfrak{a}} \in \mathcal{Z}_N(\overline{M_K}, \partial \overline{M_K}; \mathbb{R}),$$

where we view ψ as a character on the ray class group.

Limitations of the work of Kudla-Millson. The integral of $\Theta_{KM}(\tau,\varphi_f)$ along a compact cycle is a modular form of weight N whose Fourier coefficients are intersection numbers. If we replace the compact cycle by an arbitrary non-compact cycle C, then the results of Kudla and Millson do not apply and the following problem may arise. First the integral (1.0.4) might diverge. Secondly, even if the integral does converge the resulting function in τ can be non-holomorphic. This is for example what happens in [Fun02]: the Kudla-Millson theta series associated to a quadratic space of signature $(1,2)$ is integrated over the whole modular curve and the resulting integral converges to a non-holomorphic modular form. Thirdly, it is not immediate that the Fourier coefficients $\int_C \Theta_n(\varphi_f)$ can be interpreted as an intersection number between the two non-compact cycles $C_n(\varphi_f)$ and C, since a priori such a number is not well-defined; see Remark 2.2.3.

In our case, the integral of $\Theta_{KM}(\tau,\varphi_f)$ over $C\otimes\psi$ does not converge, but can be regularized by adding a parameter t^s and isolating some singular terms, as it is done in [Kud82]. Moreover, although the cycles $C\otimes\psi$ and $C_n(\varphi_f)$ are both non-compact we show that the intersection number $\left\langle C_n(\varphi_f), C\otimes\psi\right\rangle$ in M_K is well-defined, see (4.3.14). Recall that $C\otimes\psi$ is a torus of dimension N and $C_n(\varphi_f)$ of codimension N.

Eisenstein series and diagonal restriction. For a Hecke character ψ as above and a finite Schwartz function $\phi_f \in \mathscr{S}(F^2_{\mathbb{A},f})$ we can define an Eisenstein series

$$E(\tau_1,\dots,\tau_N,\phi_f,\psi) = E(\tau_1,\dots,\tau_N,\phi_f,\psi,s)\big|_{s=0}$$

by analytic continuation, where $(\tau_1,\dots,\tau_N)\in\mathbb{H}^N$. It is a Hilbert modular form of parallel weight 1. If we take $\tau=\tau_1=\cdots=\tau_N$, then the diagonal restriction

$$\tau \longmapsto E(\tau,\dots,\tau,\phi_f,\psi)$$

is a modular form of weight N.

Let l_1 and l_2 be the isotropic lines spanned by the isotropic vectors $\mathbf{e}_1 \coloneqq {}^{\mathrm{t}}(1,0)$ and $\mathbf{e}_2 \coloneqq {}^{\mathrm{t}}(0,1)$ in F^2. For a Schwartz function on $X_{\mathbb{A}_f}\simeq F^2_{\mathbb{A}_f}$ let φ_1 and φ_2 in $\mathscr{S}(F_{\mathbb{A},f})$ be the restriction of φ_f to l_1 and l_2.

Theorem B. *(Theorem 4.3.12) Let $\varphi_f\in\mathscr{S}(X_{\mathbb{A}_f})$ be any Schwartz function such that φ_1 or φ_2 vanishes. The regularized integral*

$$2^{N-1}\int_{C\otimes\psi}\Theta_{KM}(\tau,\varphi_f)$$

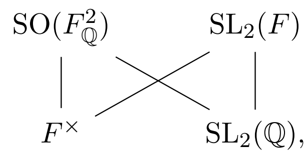
is the diagonal restriction of $E(\tau_1,\dots,\tau_N,\phi_f,\psi)$, where $\phi_f=\mathscr{F}\varphi_f$ is a partial Fourier transform *of* φ_f*. Moreover, the diagonal restriction has the Fourier expansion*

$$E(\tau,\dots,\tau,\phi_f,\psi) = \zeta_f(\varphi_1,\psi^{-1},0) + \zeta_f(\varphi_2,\psi,0) + (-1)^N 2^{N-1}\kappa \sum_{n\in\mathbb{Q}_{>0}} \langle C_n(\varphi_f), C\otimes\psi\rangle e^{2i\pi n\tau},$$

where ζ_f *is a zeta integral (see 4.1.3) and* κ *is* 1 *or* 2 *as in* (1.0.5).

The intersection takes place in M_K and is between the N-dimensional cycle $C\otimes\psi$ of dimension N and the cycle $C_n(\varphi_f)$ of codimension N (and dimension N^2-N). The constant term consists of the values at $s=0$ of the analytic continuation of two zeta functions converging on two disjoint half planes $\mathrm{Re}(s)>1$ and $\mathrm{Re}(s)<-1$. Hence the condition of vanishing of φ_1 or φ_2 is used to make sure that one of the two terms is zero and that an analytic continuation exists.

A seesaw. The theta series $\Theta_{KM}(\tau,\varphi_f)$ is a theta kernel for the dual pair $\mathrm{SL}_2(\mathbb{Q})\times\mathrm{SO}(F^2_{\mathbb{Q}})$. The cycle $C\otimes\psi$ is a tori obtained from the group embedding $F^\times\subset\mathrm{SO}(F^2_{\mathbb{Q}})$. On the other hand the diagonal restriction of the Hilbert modular form $E(\tau_1,\dots,\tau_N,\phi_f,\psi)$ comes from the diagonal embedding of $\mathrm{SL}_2(\mathbb{Q})$ in $\mathrm{SL}_2(F)$. Hence this theorem can be summarized by the following *see-saw* diagram



that relates the theta kernel $\Theta_{KM}(\tau,\varphi_f)$ to a theta kernel for $F^\times\times\mathrm{SL}_2(F)$.

Specialization to quadratic fields. In Section 4.4 we specialize to the case where F is a quadratic field, to recover the result from Darmon-Pozzi-Vonk. In this case we can identify $F^2_{\mathbb{Q}}$ with $\mathrm{Mat}_2(\mathbb{Q})$, where $\mathrm{Mat}_2(\mathbb{Q})$ is the quadratic space of 2 by 2 matrices with the quadratic form $2\det$. It is a space of signature $(2,2)$ and we already mentioned that in this case $\mathbb{D}^+$ can be indentified with $\mathbb{H}\times\mathbb{H}$. Moreover, we can choose K such that M_K is isomorphic to a product of modular curves $Y_0(p)\times Y_0(p)$.

Suppose that p is split. Each root r of D_F modulo p gives an isomorphism $\mathscr{O}\simeq\mathbb{Z}^2$, which gives two different classes $C_{\pm r}\otimes\psi$. In this setting the relative cycle

$$C_r\otimes\psi + C_{-r}\otimes\psi \in \mathcal{Z}_2\left(\overline{Y_0(p)}^2, \partial\overline{Y_0(p)}^2\right),$$

is equal to $\overline{\mathcal{Q}}(\psi)\times\overline{\mathcal{Q}}(\infty,0)$ where the boundary is given by

$$\partial\overline{Y_0(p)}^2 = \overline{Y_0(p)}\times\partial\overline{Y_0(p)} \cup \partial\overline{Y_0(p)}\times\overline{Y_0(p)}.$$

For an appropriate choice of Schwartz function $\varphi_f^{(p)}\in\mathscr{S}(\mathrm{Mat}_2(\mathbb{A}_f))$, the special cycles $C_n(\varphi_f^{(p)})$ are correspondences in $Y_0(p) \times Y_0(p)$ and we have

$$\langle C_n(\varphi_f^{(p)}), C \otimes \psi \rangle = \langle \overline{\mathcal{Q}}(0,\infty), T_n \overline{\mathcal{Q}}(\psi) \rangle.$$

Moreover, if $\phi^{(p)} \in \mathscr{S}(\mathbb{A}_F^2)$ be the partial Fourier transform of $\varphi^{(p)}$, then

$$E^{(p)}(\tau,\tau,\psi) := E\left(\tau,\tau,\phi_f^{(p)},\psi\right)$$

is the p-stabilization considered earlier. In that way we recover the result from Darmon-Pozzi-Vonk, see Corollary 4.4.12.1

Remark 1.2.1. Note that in our result there is a constant 4 in front of the non-constant Fourier coefficients, which differs from the factor 2 in *loc. cit.*. This is due to the absence of the factor κ there; see Remark 4.4.7 for more details.

Part 2

In Chapter 5 we consider the symmetric space Y_Γ associated to $\mathrm{SL}_2(K)$ where $K/\mathbb{Q}$ is an imaginary quadratic field. Let $\mathscr{O}$ be its ring of integers, that we suppose to have no non-trivial units. Let $Y_\Gamma := \Gamma \backslash \mathbb{H}_3$ where $\mathbb{H}_3$ is the hyperbolic 3-space on which $\Gamma := \mathrm{SL}_2(\mathscr{O})$ acts. It is a non-compact space and let X_Γ be the Borel-Serre compactification with boundary ∂X_Γ.

1.3 Denominators of Eisenstein cohomology

The inclusion $Y_\Gamma \hookrightarrow X_\Gamma$ is a homotopy equivalence, hence $H^1(Y_\Gamma;\mathbb{C})$ can be identified with $H^1(X_\Gamma;\mathbb{C})$. By restriction to the boundary we get a map

$$\mathrm{res}\colon H^1(Y_\Gamma;\mathbb{C}) \longrightarrow H^1(\partial X_\Gamma;\mathbb{C}),$$

whose kernel is the interior (or cuspidal) cohomology that we denote by $H^1_!(Y_\Gamma;\mathbb{C})$. It can be identified with the image of the compactly supported cohomology $H^1_c(Y_\Gamma;\mathbb{C})$ inside $H^1(Y_\Gamma;\mathbb{C})$. The Eisenstein cohomology $H^1_{\mathrm{Eis}}(Y_\Gamma;\mathbb{C})$ is a complement to the cuspidal cohomology, so that we have a splitting

$$H^1(Y_\Gamma;\mathbb{C}) = H^1_!(Y_\Gamma;\mathbb{C}) \oplus H^1_{\mathrm{Eis}}(Y_\Gamma;\mathbb{C}).$$

Let ψ be an unramified algebraic Hecke character of infinity type z^2, taking values in a finite extension F_ψ of K. The Eisenstein cohomology is the image of $\mathrm{Im}(\mathrm{res}) \subset H^1(\partial X_\Gamma;\mathbb{C})$ by Harder's

Eisenstein map

$$\mathrm{Eis}_\psi \colon \mathrm{Im}(\mathrm{res}) \longrightarrow H^1(Y_\Gamma; \mathbb{C}).$$

In [BCG20; BCG21] Bergeron-Charollois-Garcia use the Mathai-Quillen form to construct a Γ-invariant differential form

$$E_\psi \in \Omega^1(\mathbb{H}_3; \mathbb{C})^\Gamma$$

on $\mathbb{H}_3$ that represents an Eisenstein class in the cohomology $H^1_{\mathrm{Eis}}(Y_\Gamma; \mathbb{C})$. More generally, they define this form on the symmetric space associated to $\mathrm{SL}_N(K)$ and with non-trivial coefficients. In the case that we consider ($N = 2$ and no coefficients) these differential forms already appear in the work of Ito [Ito87]. This form defines a cocycle $E_\psi \in H^1(\Gamma, \mathbb{C})$ by

$$E_\psi(\gamma) = \int_{u_0}^{\gamma u_0} E_\psi$$

where u_0 is a point in $\mathbb{H}_3$. For a lattice L in K define the Sczech cocycle $\Phi_L \colon \Gamma \longrightarrow \mathbb{C}$ by

$$\Phi_L \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{cases} I\left(\frac{a+d}{c}\right) E_2(L) - D(a,c,L) & \text{if} \quad c \neq 0, \\ I\left(\frac{b}{d}\right) E_2(L) & \text{if} \quad c = 0, \end{cases}$$

where $E_2(L)$ is an Eisenstein series and $D(a, c, L)$ a Dedekind sum; see Section 5.1 for the definitions. The two cocycles are related by

$$E_\psi = \sum_{\mathfrak{a} \in \mathrm{Cl}(K)} \psi(\mathfrak{a})^{-1} \Phi_{\mathfrak{a}}. \tag{1.3.1}$$

The equality is proved by using the idea of [BCG21] to move the path of integration $[u_0, \gamma u_0]$ to infinity. More precisely, choose a cusp v of Y_Γ and let $[v, \gamma^{-1} v]$ be the modular symbol joining the two cusps v and $\gamma^{-1} v$. There is a homotopy between $[u_0, \gamma u_0]$ and $[v, \gamma^{-1} v]$; see Figure 5.3. The integral along the modular symbol $[v, \gamma^{-1} v]$ gives the Dedekind sum, whereas the term $I\left(\frac{a+d}{c}\right) E_2(\mathfrak{a})$ is a contribution from the cusps. Note that this formula already appears in the work of Ito [Ito87, Theorem. 3].

We can use Ito's formula to deduce integrality results about the Eisenstein cohomology. Sczech proved that the cocycle $2\Phi_L$ is integral if the Eisenstein series $g_2(L)$ and $g_3(L)$ are algebraic integers, see Proposition 5.3.4. Let $L^{\mathrm{alg}}(\psi, 0)$ be a normalization of $L(\psi, 0)$ so that it is an algebraic number. Let K_ψ be an extension of K containing the image of Φ_L, the value $L^{\mathrm{alg}}(\psi, 0)$ and the field F_ψ; let $\mathscr{O}_\psi$ be the ring of integers of K_ψ. Harder showed that the Eis_ψ map is rational *i.e.* it restricts to a map

$$\mathrm{Eis}_\psi\colon\ \mathrm{Im}(\mathrm{res}) \cap H^1(\partial X_\Gamma; K_\psi) \longrightarrow H^1(Y_\Gamma; K_\psi).$$

We are interested in the restriction to the integral coefficients:

$$\mathrm{Eis}_\psi\colon\ \mathrm{Im}(\mathrm{res}) \cap \widetilde{H}^1(\partial X_\Gamma; \mathscr{O}_\psi) \longrightarrow H^1(Y_\Gamma; K_\psi), \tag{1.3.2}$$

where $\widetilde{H}^1(-;\mathscr{O}_\psi)$ denotes the torsion-free part of the cohomology and is identified with

$$\mathrm{Im}\left(H^1(-;\mathscr{O}_\psi) \longrightarrow H^1(-;\mathbb{C})\right).$$

We may wonder if the Eisenstein map is also integral, in the sense that the image of (1.3.2) lies in $\widetilde{H}^1(Y_\Gamma;\mathscr{O}_\psi)$. We define the denominator $\delta_{K_\psi}(\mathrm{Eis}_\psi)$ of the Eisenstein cohomology to be the fractional ideal such that the image of the map (1.3.2) multiplied by the denominator lies in $\widetilde{H}^1(Y_\Gamma;\mathscr{O}_\psi)$. In particular if the denominator is $\mathscr{O}_\psi$ then the Eisenstein map is integral. A conjecture of Harder (in a much more general setting) relates the denominator of Eisenstein cohomology to special values of L-functions. We prove the following.

Theorem C. *(Theorem 5.3.5) Suppose that K has class number 1. We have the upper bound (in the sense of divisibility) on the denominator*

$$2L^{\mathrm{alg}}(\psi,0)\mathscr{O}_\psi \subset \delta_{K_\psi}(\mathrm{Eis}_\psi).$$

On the other hand, Berger [Ber08] gives a lower bound of the denominator, so that in some cases we get the exact denominator; see Subsection 5.3.4.

For the moment we only prove Theorem 5.3.5 for class number 1 and without coefficients. The more general case of arbitrary class number and arbitrary coefficients should be accessible with the same idea. Moreover, in principle it should also be possible to prove similar results for $\mathrm{SL}_N(K)$; see remark below.

Remark 1.3.1. In their work, Bergeron-Charollois-Garcia [BCG21, Proposition. 3.1] use the Eisenstein class E_ψ for $\mathrm{SL}_N(K)$ to show a relation between a Dedekind sum and an integral along a closed geodesic on the space associated to $\mathrm{SL}_N(K)$. They consider a smoothed Eisenstein series in order to make it rapidly decreasing at a cusp v. Hence they obtain a formula similar to (1.3.1) but without any contributions from the cusps. In particular, by generalizing the proof of (1.3.1) to $\mathrm{SL}_N(K)$ one could obtain a generalization of Ito's formula and of [BCG21]. In particular, this could also give upper bounds on the denominator of the Eisenstein cohomology for $\mathrm{SL}_N(K)$.

Chapter 2

Background

2.1 Notations and generalities

In this section we fix some notations and collect a few facts that we will need.

2.1.1 Intersection numbers

Let M be an arbitrary oriented Riemannian manifold of dimension m, let q_z denote the Riemannian metric and $o(T_zM) \in \bigwedge^m T_zM$ the orientation at a point $z \in M$. Let M_1 and M_2 be two immersed oriented submanifolds of dimensions m_1 and m_2 such that $m_1 + m_2 = m$, with orientations $o(T_zM_1)$ and $o(T_zM_2)$. Let z be a point in the intersection $M_1 \cap M_2$. We say that the intersection is transversal if $T_zM = T_zM_1 \oplus T_zM_2$, equivalently if $T_zM_1 \cap T_zM_2 = 0$. In this case $o(T_zM_1) \wedge o(T_zM_2)$ is a multiple of $o(T_zM)$ and we define

$$I_z(M_1, M_2) := \operatorname{sgn}\left(\frac{o(T_zM_1) \wedge o(T_zM_2)}{o(T_zM)}\right).$$

Furthermore, if $M_1 \cap M_2$ is finite we define the *topological intersection number*

$$\langle M_1, M_2 \rangle := \sum_{z \in M_1 \cap M_2} I_z(M_1, M_2).$$

Note that if one of the two submanifolds is compact, then the hypothesis is satisfied.

Viewing the Riemannian metric q_z as a bilinear pairing on T_zM we get a bilinear pairing

$$\begin{aligned} \bigwedge^k T_zM \otimes \bigwedge^k T_zM &\longrightarrow \mathbb{R} \\ v_1 \wedge \cdots \wedge v_k \otimes w_1 \wedge \cdots \wedge w_k &\longmapsto \det q_z(v_i, w_j)_{ij} \end{aligned}$$

for every $k \geq 0$. Let $N_zM_1 = (T_zM_1)^\perp$ be the normal space, which is the orthogonal complement of T_zM_1 with respect to q_z. We fix an orientation $o(N_zM_1)$ of N_zM_1 via the rule

$$o(T_zM_1) \wedge o(N_zM_1) = o(T_zM).$$

Lemma 2.1.1. *The intersection number is given by*

$$I_z(M_1, M_2) = \operatorname{sgn} q_z\big(o(T_zM_2), o(N_zM_1)\big).$$

Proof. Let $p_z \colon \bigwedge^k T_zM \longrightarrow \bigwedge^k N_zM_1$ be the orthogonal projection for any $k \geq 0$. It induces an isomorphism between N_zM_1 and T_zM_2 if and only if

$$T_zM_2 \cap \ker(p_z) = T_zM_2 \cap T_zM_1 = 0,$$

that is precisely when M_1 and M_2 are transversal. Let $v_1, \ldots, v_{m_1}$ be a basis of T_zM_1 and $w_1, \ldots, w_{m_2}$ be an orthogonal basis of T_zM_2. Let $p_z(w_1), \ldots, p_z(w_{m_2})$ be the orthogonal projection of this basis, giving a basis of N_zM_1. Suppose that the bases are ordered such that $o(T_zM) = v_1 \wedge \cdots \wedge v_{m_1}$ and $o(T_zM_2) = w_1 \wedge \cdots \wedge w_{m_2}$. A direct computation shows that since the basis is orthogonal we have

$$p_z(o(T_zM_2)) = p_z(w_1 \wedge \cdots \wedge w_{m_2}) = p_z(w_1) \wedge \cdots \wedge p_z(w_{m_2}). \tag{2.1.1}$$

On the hand, since $\bigwedge^{m_2} N_zM_1$ is one dimensional we have

$$p_z(o(T_zM_2) = \alpha_z o(N_zM_1)$$

where

$$\alpha_z = \frac{q_z\big(o(T_zM_2), o(N_zM_1)\big)}{q_z\big(o(N_zM_1), o(N_zM_1)\big)}.$$

Writing $p_z(w_k) = w_k + u$ some $u \in T_zM_1$ we see that

$$o(T_zM_1) \wedge p_z(w_k) = o(T_zM_1) \wedge w_k$$

for every k. Hence

$$\begin{aligned}
o(T_zM_1) \wedge o(T_zM_2) &= o(T_zM_1) \wedge w_1 \wedge \cdots \wedge w_{m_2} \\
&= o(T_zM_1) \wedge p_z(w_1) \wedge \cdots \wedge p_z(w_{m_2}) \\
&= o(T_zM_1) \wedge p_z(w_1 \wedge \cdots \wedge w_{m_2}) \\
&= \alpha_z o(T_zM_1) \wedge o(N_zM_1) \\
&= \alpha_z o(T_zM).
\end{aligned}$$

Thus the intersection at z is given by

$$I_z(M_1, M_2) = \operatorname{sgn} \alpha_z = \operatorname{sgn} q_z\big(o(T_zM_2), o(N_zM_1)\big),$$

since $q_z\big(o(N_zM_1), o(N_zM_1)\big) > 0$.

Note that when M_1 and M_2 are not transversal, then for one of the basis elements w_k we have $p_z(w_k) = 0$. Hence by (2.1.1) we have $\alpha_z = 0$. □

2.1.2 Poincaré duals

Let M be an m-dimensional oriented real manifold without boundary. The integration map yields a non-degenerate pairing [BT82, Theorem. 5.11]

$$H^q(M) \otimes_{\mathbb{R}} H_c^{m-q}(M) \longrightarrow \mathbb{R}$$
$$[\omega] \otimes [\eta] \longmapsto \int_M \omega \wedge \eta,$$

where H_c denotes the cohomology of compactly supported forms. This yields the isomorphism

$$H_c^{m-q}(M)^\vee \cong H^q(M) \tag{2.1.2}$$

where $H_c^{m-q}(M)^\vee = \mathrm{Hom}_{\mathbb{R}}(H_c^{m-q}(M), \mathbb{R})$ is the space of functionals. If C is an immersed submanifold of M of codimension q, then it defines a linear functional on $H_c^{m-q}(M)$ by

$$\omega \longmapsto \int_C \omega.$$

By the isomorphism (2.1.2) there is a unique cohomology class $\mathrm{PD}(C) \in H^q(M)$ representing this functional *i.e.*

$$\int_M \omega \wedge \mathrm{PD}(C) = \int_C \omega$$

for every $[\omega] \in H_c^{m-q}(M)$. We call $\mathrm{PD}(C)$ *the Poincaré dual class to* C, and any differential form representing the cohomology class $\mathrm{PD}(C)$ *a Poincaré dual form to* C. If C is compact, then $\mathrm{PD}(C)$ is compactly supported. Let C and C' be immersed submanifolds of complementary dimensions and let one of them be compact. Then

$$\int_C \mathrm{PD}(C') = \int_{M_K} \mathrm{PD}(C') \wedge \mathrm{PD}(C) = \langle C', C \rangle,$$

where the right hand side is the topological intersection numbers.

2.1.3 Thom forms

Let $\pi\colon \mathcal{E} \longrightarrow M$ be an oriented vector bundle of rank q over a connected smooth manifold M. Suppose that the vector bundle is Euclidean *i.e.* that it has a metric in every fiber that varies smoothly over the base. Let $\mathrm{D}(\mathcal{E})$ be the closed disk bundle. If we have a closed $(q+i)$-form on $\mathcal{E}$ whose support is contained in $\mathrm{D}(\mathcal{E})$, then it has compact support in the fiber and represents a class in $H^{q+i}(\mathcal{E}, \mathcal{E} - \mathrm{D}(\mathcal{E}))$. Fiber integration induces an isomorphism on the level of cohomology

$$\begin{aligned} \mathrm{Th}\colon H^{q+i}(\mathcal{E}, \mathcal{E} - \mathrm{D}(\mathcal{E})) &\longrightarrow H^i(M) \\ [\omega] &\longmapsto \int_{\text{fiber}} \omega \end{aligned}$$

known as the *Thom isomorphism* [BT82, Theorem. 6.17]. In particular $H^0(M) = \mathbb{R}$, and we call the preimage of 1

$$\mathrm{Th}(\mathcal{E}) \coloneqq \mathrm{Th}^{-1}(1) \in H^q(\mathcal{E}, \mathcal{E} - \mathrm{D}(\mathcal{E}))$$

the *Thom class*. Any differential form representating this class is called a *Thom form*. In particular every closed q-form on $\mathcal{E}$ that has compact support in every fiber and whose integral along every fiber is 1 is a Thom form. One can also show that the Thom class is the Poincaré dual class of the zero section $\mathcal{E}_0$ in $\mathcal{E}$.

Let $\omega \in \Omega^j(\mathcal{E})$ be a form on the bundle and let ω_z be its restriction to a fiber $\mathcal{E}_z = \pi^{-1}(z)$ for some $z \in M$. After identifying $\mathcal{E}_z$ with $\mathbb{R}^q$ we get a form $\omega_z \in C^\infty(\mathbb{R}^q) \otimes \wedge^j(\mathbb{R}^q)^\vee$. We say that ω is *rapidly decreasing in the fiber* if $\omega_z \in \mathscr{S}(\mathbb{R}^q) \otimes \wedge^j(\mathbb{R}^q)^\vee$ for all $z \in M$ and we write $\Omega^j_{\mathrm{rd}}(\mathcal{E})$ for the space of such forms. Let $H_{\mathrm{rd}}(\mathcal{E})$ the cohomology of the complex of rapidly decreasing forms. The map

$$\begin{aligned} h\colon \mathcal{E} &\longrightarrow \mathcal{E} \\ \mathbf{y} &\longrightarrow \frac{\mathbf{y}}{\sqrt{1 - \|\mathbf{y}\|^2}} \end{aligned}$$

is a diffeomorphism from the open disk bundle $\mathrm{D}(\mathcal{E})^\circ$ onto $\mathcal{E}$. It induces an isomorphism by pullback

$$h^*\colon H_{\mathrm{rd}}(\mathcal{E}) \longrightarrow H(\mathcal{E}, \mathcal{E} - \mathrm{D}(\mathcal{E})),$$

which commutes with the fiber integration. Hence we have an alternative version of the Thom isomorphism

$$H^{q+i}_{\mathrm{rd}}(\mathcal{E}) \longrightarrow H^i(M).$$

After choosing a connection ∇ compatible with the metric, a construction of Mathai and Quillen [MQ86] produces a canonical Thom form

$$U_{MQ} \in \Omega^q_{\mathrm{rd}}(\mathcal{E}).$$

We will discuss this construction in Chapter 3.

2.1.4 Local fields

Let $F = \mathbb{Q}(\lambda)$ be a totally real field of degree N and let f_λ be the minimal polynomial of λ. Let w be a place of F and v a place of $\mathbb{Q}$. Over $\mathbb{Q}_v$ the (separable) minimal polynomial f_λ of λ splits into irreducible factors

$$f_\lambda(x) = \prod_{w|v} f_w(x)$$

where the factors correspond to the places w dividing v [Neu99, Proposition. 8.2, p. 163]. For every w let $\lambda_w \in \overline{\mathbb{Q}}_v$ be a root of f_w and let $F_w \coloneqq \mathbb{Q}_v(\lambda_w)$. Let $|x|_w \coloneqq \sqrt[n_w]{|\operatorname{N}(x)|_v}$ be the valuation on F_w that extends $|\cdot|_v$, where $n_w \coloneqq \deg(f_w)$ is the degree of F_w over $\mathbb{Q}_v$. If w is finite we denote by $\mathscr{O}_w$ its ring of integers, $\mathfrak{m}_w$ its unique maximal ideal, π_w a uniformizer and q_w the cardinality of the residue field $\mathscr{O}_w/\mathfrak{m}_w$. If $\mathfrak{p}$ is the prime ideal corresponding to a finite place w we will occasionally replace the index w by $\mathfrak{p}$ and write $F_\mathfrak{p}, \mathscr{O}_\mathfrak{p}, |\cdot|_\mathfrak{p}, \dots$

If $v = p$ is finite, then by Hensel's Lemma [Neu99, Proposition. 4.6, p. 129] there is an irreducible polynomial $P_w \in \mathbb{Z}[x]$ such that

$$f_w(x) \equiv P_w(x)^{e_w} \pmod{p}.$$

Hence the minimal polynomial of λ splits as

$$f_\lambda(x) \equiv \prod_{w|v} P_w(x)^{e_w} \pmod{p}.$$

If $\mathscr{O} = \mathbb{Z}[\lambda]$, then p splits as

$$p\mathscr{O} = \prod_{w|v} \mathfrak{p}_w^{e_w}, \tag{2.1.3}$$

where $\mathfrak{p}_w = P_w(\lambda)\mathbb{Z} + p\mathbb{Z}$; see [Neu99, Proposition. 8.3, p. 48].

2.1.5 Adèles

Let $\mathbb{A} \coloneqq \prod'_v \mathbb{Q}_v$ be the ring of adèles $\mathbb{A}$ over $\mathbb{Q}$, where the symbol $'$ denotes the restricted product with respect to $\widehat{\mathbb{Z}} \coloneqq \prod_p \mathbb{Z}_p$. This means that an adèle $x = (x_\infty, x_2, x_3, x_5, \dots) \in \mathbb{A}$ is a tuple of elements $x_v \in \mathbb{Q}_v$ such that $x_p \in \mathbb{Z}_p$ for almost all primes p. The topology on this restricted product has a basis of open set sets of the form

$$U_\infty \times \prod_{p\in S} U_p \times \prod_{p\notin S} \mathbb{Z}_p$$

where S is a finite set of primes, and U_v is open in $\mathbb{Q}_v$ for $v \in S$. With this topology it is a locally compact group. The finite part of the adèles is $\mathbb{A}_f := \prod_p \mathbb{Q}_p$. Similarly, we define the adèles over a number field F by $\mathbb{A}_F := \prod'_w F_w$ which is the restricted product with respect to $\widehat{\mathscr{O}} := \prod_w \mathscr{O}_w$.

Let H be a linear algebraic group over $\mathbb{Q}$, that we view as a $\mathbb{Q}$-closed subgroup of GL_n for some n. Let $H(\mathbb{Z}_v) = H(\mathbb{Q}_v) \cap \mathrm{GL}_n(\mathbb{Z}_v)$ and $H(\mathbb{A}) := \prod'_v H(\mathbb{Q}_v)$ be the restricted product with respect to $H(\widehat{\mathbb{Z}}) = \prod_v H(\mathbb{Z}_v)$. The topology is generated by open sets of the form

$$U_\infty \times \prod_{p \in S} U_p \times \prod_{p \notin S} H(\mathbb{Z}_p)$$

where S is a finite set of primes, and U_v is open in $H(\mathbb{Q}_v)$ for $v \in S$. One example that we will consider is the (multiplicative) group of idèles $\mathbb{A}^\times := \mathrm{GL}_1(\mathbb{A})$, the units of $\mathbb{A}$. Similarly we define $H(\mathbb{A}_F)$ if H is defined over F.

Let $X_\mathbb{Q}$ be a $\mathbb{Q}$-vector space and $X_{\mathbb{Q}_v} = X_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{Q}_v$ for every place v of $\mathbb{Q}$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be a basis of $X_\mathbb{Q}$ and $X_\mathbb{Z} = \mathbb{Z}\mathbf{x}_1 + \cdots + \mathbb{Z}\mathbf{x}_n$ the $\mathbb{Z}$-span of the basis; it is a lattice in $X_\mathbb{Q}$. Let $X_{\widehat{\mathbb{Z}}} := \prod_v X_{\mathbb{Z}_v}$ where $X_{\mathbb{Z}_v} = X_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}_v$. Then we define $X_\mathbb{A} := \prod'_v X_{\mathbb{Q}_v}$ to be the restricted product with respect to $X_{\widehat{\mathbb{Z}}}$. Similarly if X is an F-vector space we can define $X_{\mathbb{A}_F}$.

Let $\mathscr{S}(X_{\mathbb{Q}_v})$ be the space of usual Schwartz functions on $X_{\mathbb{Q}_v} \simeq \mathbb{R}^n$ if $v = \infty$ and the space of compactly supported and locally constant $\mathbb{C}$-valued functions on $X_{\mathbb{Q}_p}$ if $v = p$. In particular the characteristic function $\mathbf{1}_{X_{\mathbb{Z}_p}}$ lies in $\mathscr{S}(X_{\mathbb{Q}_p})$. We define the Schwartz space $\mathscr{S}(X_\mathbb{A})$ to be the space of finite linear combinations of functions $\varphi = \prod_v \varphi_v$ where $\varphi_v \in \mathscr{S}(X_{\mathbb{Q}_v})$ and $\varphi_p = \mathbf{1}_{X_{\mathbb{Z}_p}}$ for almost all primes. In general, if we identify $X_{\mathbb{Q}_p}$ with $\mathbb{Q}_p^n$ then any Schwartz function φ_p is of the shape

$$\varphi_p = \sum_{i=1}^N c_i \mathbf{1}_{B_{p^{-k_i}}(\mathbf{x}_i)}$$

where $B_{p^{-k_i}}(\mathbf{x}_i)$ is the ball of radius p^{-k_i} and center $\mathbf{x}_i$. Here the norm on $\mathbb{Q}_p^n$ is given by taking the maximum of the p-adic norms of the components. Again this definition of Schwartz functions extends analogously to an F-vector space.

For more on adèles and Schwartz-Bruhat functions see [PR94, Chapter. 5, page. 243], [Wei95], [Bum97, Section. 3.1 , Section. 3.3] or more generally Tate's thesis.

2.1.6 Haar measures

Let F be a totally real number field and F_w the completion at a place w. We identify F_w with its dual $F_w^\vee$ by the pairing $\mathrm{tr}_{F_w/\mathbb{Q}_v}(xy)$ as in (4.1.1). We fix the additive complex character $\chi_{\mathbb{Q}_v}$ on $\mathbb{Q}_v$

defined by

$$\chi_{\mathbb{Q}_v}(x) \coloneqq \begin{cases} e^{2i\pi x} & \text{if } v = \infty \\ e^{-2i\pi \{x\}_p} & \text{if } v = p, \end{cases}$$

where $\{x\}_p$ is the fractional part of x in $\mathbb{Q}_p$. Let k be a finite algebra extension of $\mathbb{Q}_v$. We can extend this character to a character χ_k on k by setting $\chi_k \coloneqq \chi_{\mathbb{Q}_v} \circ \mathrm{tr}_{k|\mathbb{Q}_v}$.

There is a unique choice of Haar measure dx_w on F_w which is self dual with respect to χ_{F_w}. This is the Haar measure normalized such that $\mathrm{vol}(\mathfrak{d}_w^{-1}) = 1$, where

$$\mathfrak{d}_w^{-1} = \left\{ x \in F_w \middle| \ \mathrm{tr}_{F_w/\mathbb{Q}_v}(xy) \in \mathbb{Z}_v \, \text{for all}\, y \in \mathscr{O}_w \right\}$$

is the inverse different ideal. This is equivalent to $\mathrm{vol}(\mathscr{O}_w) = \sqrt{\mathrm{N}(\mathfrak{d}_w)}$. We obtain a measure $dx \coloneqq \prod_w dx_w$ on $\mathbb{A}_F$ which satisfies $\mathrm{vol}(\widehat{\mathscr{O}}) = d_F^{-\frac{1}{2}}$. For a function $\Phi_w \in \mathscr{S}(F_w)$ let $\Phi_w^\vee \in \mathscr{S}(F_w)$ be the Fourier transform defined by

$$\Phi_w^\vee(t) \coloneqq \int_{\mathbb{A}_F} \Phi_w(u)\chi_w(-tu)du, \tag{2.1.4}$$

where du is the self-dual measure. On $\mathbb{A}_F^\times$ we define the Haar measure

$$dt_w^\times \coloneqq m_w \frac{dt_w}{|t_w|_w},$$

where $m_w = 1$ if w is archimedean. If m_w is non-archimedean then the m_w will be choosen such that $\mathrm{vol}^\times(\widehat{U}(\mathfrak{f})) = 1$ where $\mathfrak{f}$ will be the conductor of ψ.

2.1.7 Class groups

Let F be a number field of degree N. For an ideal $\mathfrak{f} \subset \mathscr{O}$ let $\mathcal{I}_\mathfrak{f}(F)$ be the set of fractional ideals in F coprime to $\mathfrak{f}$, and let $P_\mathfrak{f}(F) \subset \mathcal{I}_\mathfrak{f}(F)$ be the set of principal ideals. Let $P_\mathfrak{f}(F)^+ \subset P_\mathfrak{f}(F)$ be the set of principal ideals whose generator is positive at all real places. Let $\mathrm{Cl}(F) \coloneqq \mathcal{I}_\mathscr{O}(F)/P_\mathscr{O}(F)$ be the class group and $\mathrm{Cl}(F)^+ \coloneqq \mathcal{I}_\mathscr{O}(F)/P_\mathscr{O}(F)^+$ the narrow class group, whose cardinals we denote by h and h^+. We have an exact sequence

$$1 \longrightarrow \mathscr{O}^\times/\mathscr{O}^{\times,+} \longrightarrow F^\times/F^{\times,+} \longrightarrow \mathrm{Cl}(F)^+ \longrightarrow \mathrm{Cl}(F) \longrightarrow 1.$$

If F has r real embedings and s pairs of complex embeddings ($N = r + 2s$) then $[F^\times : F^{\times,+}] = 2^r$ and

$$h^+ = h\frac{2^r}{[\mathscr{O}^\times : \mathscr{O}^{\times,+}]}. \tag{2.1.5}$$

We define the ray class groups

$$\begin{aligned} \mathrm{Cl}_{\mathfrak{f}}(F) &:= \mathcal{I}_{\mathfrak{f}}(F)/P_{\mathfrak{f}}(F) \\ \mathrm{Cl}_{\mathfrak{f}}(F)^+ &:= \mathcal{I}_{\mathfrak{f}}(F)/P_{\mathfrak{f}}(F)^+. \end{aligned}$$

Let now F be totally real. Define

$$U_w(\mathfrak{f}) := \begin{cases} \mathscr{O}_w^\times & \text{if } w(\mathfrak{f}) = 0 \\ 1 + \mathfrak{p}^{w(\mathfrak{f})} & \text{if } w(\mathfrak{f}) > 0. \end{cases}$$

and

$$\widehat{U}(\mathfrak{f}) := \prod_{w<\infty} U_w(\mathfrak{f})$$

where the product is taken over all finite places w of F. Define the restricted product with respect to the open compact subgroups $\mathscr{O}_w^\times$

$$\widehat{V}(\mathfrak{f}) := \prod_{w(\mathfrak{f})>0} U_w(\mathfrak{f}) \times {\prod_{w(\mathfrak{f})=0}}' F_w^\times \supset \widehat{U}(\mathfrak{f}).$$

For any $t \in \mathbb{A}_{F,f}^\times$ it follows from the approximation theorem [Neu99, Theorem. 3.4, page. 117] that we can find $\alpha \in F^\times$ such that

$$\begin{aligned} &\alpha t_w \equiv 1 \mod \mathfrak{m}_w^{w(\mathfrak{f})} && \text{if } w(\mathfrak{f}) > 0, \\ &|\alpha|_\sigma > 0 && \text{if } \sigma \text{ archimedean.} \end{aligned}$$

Hence we have $\mathbb{A}_{F,f}^\times = F^{\times,+}\widehat{V}(\mathfrak{f})$. Let us write an element t in $\mathbb{A}_{F,f}^\times$ as $\lambda(t)u(t)$ with $\lambda(t)$ in $F^{\times,+}$ and $u(t)$ in $\widehat{V}(\mathfrak{f})$. The map

$$\begin{aligned} F^{\times,+}\backslash\mathbb{A}_{F,f}^\times &\longrightarrow \widehat{V}(\mathfrak{f}) \cap F^{\times,+}\backslash\widehat{V}(\mathfrak{f}) \\ F^{\times,+}t &\longmapsto \widehat{V}(\mathfrak{f}) \cap F^{\times,+}u(t) \end{aligned}$$

is a well defined group isomorphism. Hence we also get an isomorphism

$$F^{\times,+}\backslash\mathbb{A}_{F,f}^{\times}/\widehat{U}(\mathfrak{f}) \longrightarrow \widehat{V}(\mathfrak{f}) \cap F^{\times,+}\backslash\widehat{V}(\mathfrak{f})/\widehat{U}(\mathfrak{f}). \tag{2.1.6}$$

For an ideal $\mathfrak{a} \in \mathcal{I}_{\mathfrak{f}}(F)$ we define $t_{\mathfrak{a}} \in \mathbb{A}_{F,f}^{\times}$ by

$$(t_{\mathfrak{a}})_w = \begin{cases} \pi_w^{w(\mathfrak{a})} & \text{if } w(\mathfrak{f}) = 0 \\ 1 & \text{if } w(\mathfrak{f}) > 0. \end{cases}$$

This depends on a choice of uniformizer π_w but yields a well-defined group isomorphism

$$\begin{aligned} \mathrm{Cl}_{\mathfrak{f}}(F)^{+} &\longrightarrow F^{\times,+} \cap \widehat{V}(\mathfrak{f})\backslash\widehat{V}(\mathfrak{f})/\widehat{U}(\mathfrak{f}) \\ [\mathfrak{a}] &\longmapsto F^{\times,+} \cap \widehat{V}(\mathfrak{f}) t_{\mathfrak{a}} \widehat{U}(\mathfrak{f}). \end{aligned} \tag{2.1.7}$$

The composition of the maps (2.1.6) and (2.1.7) gives an isomorphism

$$\mathrm{Cl}_{\mathfrak{f}}(F)^{+} \longrightarrow F^{\times,+}\backslash\mathbb{A}_{F,f}^{\times}/\widehat{U}(\mathfrak{f}).$$

2.1.8 Hecke characters and L-functions

Let F be a number field. Let $\mathbb{A}_F^{\times}$ be the idèles and

$$\psi\colon F^{\times}\backslash\mathbb{A}_F^{\times} \longrightarrow \mathbb{C}^{\times}$$

be a finite order Hecke character. We can write $\psi = \psi_\infty \otimes \psi_f$ where $\psi_f\colon F^{\times}\backslash\mathbb{A}_{F,f}^{\times} \to \mathbb{C}^{\times}$. The conductor of ψ is the largest ideal $\mathfrak{f} \subset \mathscr{O}$ such that ψ_f is trivial on

$$\widehat{U}(\mathfrak{f}) = \prod_{w}{}' U_w(\mathfrak{f}).$$

We say that ψ is *unramified* if the conductor is $\mathscr{O}$ *i.e.* ψ_f is trivial on $\widehat{\mathscr{O}}^{\times}$. At an archimedean places ψ_σ is given by

$$\psi_\sigma(t) = |t_\sigma|^{a_\sigma}\left(\frac{t_\sigma}{|t_\sigma|}\right)^{b_\sigma}$$

where $a_\sigma \in \mathbb{C}$ and

$$\begin{cases} b_\sigma \in \{0,1\} & \text{if } \sigma \text{ is real} \\ b_\sigma \in \mathbb{Z} & \text{if } \sigma \text{ is complex.} \end{cases}$$

In the case where F is totally real and ψ is unitary (so $a_\sigma = 0$) we say that ψ is *totally odd* if $b_\sigma = 1$ at all places.

Remark 2.1.1. The existence of totally odd characters implies that F does not contain a unit $\kappa \in \mathscr{O}^\times$ of negative norm. Suppose it does, then in the narrow class group we have $(\kappa) = \mathscr{O} = 1$. Hence if ψ were totally odd we would have

$$1 = \psi_f((\kappa)) = \psi_\infty(\kappa) = \prod_\sigma \operatorname{sgn}(\kappa^\sigma) = \frac{\mathrm{N}_{F/\mathbb{Q}}(\kappa)}{|\,\mathrm{N}_{F/\mathbb{Q}}(\kappa)|} = -1. \tag{2.1.8}$$

The finite part of a Hecke character of conductor $\mathfrak{f}$ can also be seen as a character

$$\begin{aligned} \psi\colon \mathcal{I}(\mathfrak{f}) &\longrightarrow \mathbb{C}^\times \\ \mathfrak{a} &\longmapsto \psi(1, \mathfrak{a}) \end{aligned}$$

such that at principal ideals in $\mathcal{P}^+_{\mathfrak{f}}(F)$ we have

$$\psi((\alpha)) = \prod_\sigma |\sigma(\alpha)|^{a_\sigma} \left(\frac{\sigma(\alpha)}{|\sigma(\alpha)|}\right)^{b_\sigma},$$

where the product is taken over all real places and pairs of complex embeddings. To such a character we can attach an L-function

$$L^{\mathfrak{f}}(\psi, s) := \sum_{\substack{0 \neq \mathfrak{a} \subseteq \mathscr{O} \\ \gcd(\mathfrak{f}, \mathfrak{a}) = 1}} \frac{\psi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s}.$$

It converges for $\mathrm{Re}(s) > 1$, admits a meromorphic continuation and a functional equation. Moreover, it has the Euler product

$$L^{\mathfrak{b}}(\psi, s) = \prod_{\substack{0 \neq \mathfrak{p} \subset \mathscr{O} \text{ prime} \\ \gcd(\mathfrak{p}, \mathfrak{b}) = 1}} L_{\mathfrak{p}}(\psi, s)$$

where the local factors are $L_{\mathfrak{p}}(\psi, s) := (1 - \psi(\mathfrak{p})\,\mathrm{N}(\mathfrak{p})^{-s})^{-1}$. For more on L-functions we refer to [Iwa19] and [Bum97].

2.1.9 Theta kernels

The Weil representation. Let L be a number field (we will consider $L = \mathbb{Q}$ or $L = F$ totally real) and (X_L, Q) be a $2m$-dimensional quadratic space over L. Let (W_L, B) be the symplectic space $W_L = X_L \oplus X_L$ with the symplectic form

$$B\left(\begin{pmatrix}\mathbf{x}\\ \mathbf{x}'\end{pmatrix}, \begin{pmatrix}\mathbf{y}\\ \mathbf{y}'\end{pmatrix}\right) = Q(\mathbf{x},\mathbf{y}') - Q(\mathbf{x}',\mathbf{y}).$$

Let k be the completion of L at some place, let $X_k := X \otimes_L k$ and $W_k := W_L \otimes_L k$. Let $\chi_k\colon k^\times \longrightarrow U(1)$ be the additive character defined in Subsection 2.1.6. Let

$$g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \in \mathrm{Sp}(W_k) \subset \mathrm{GL}(W_k)$$

with g_1 and g_4 in $\mathrm{GL}(X_k)$, and g_2 and g_3 in $\mathrm{End}(X_k)$. For $\varphi \in \mathscr{S}(X_k)$ we define the *local (projective) Weil representation* [MVW87, p. 40]

$$\omega\colon\ \mathrm{Sp}(W_k) \longrightarrow \mathcal{U}(\mathscr{S}(X_k))$$

by the operator

$$\begin{aligned}\omega(g)\varphi(\mathbf{x}) := \int_{X_k/\ker(g_3)} & \varphi({}^*g_1\mathbf{x} + {}^*g_3\mathbf{y}) \\ & \times \chi_k\left(\frac{1}{2}Q({}^*g_1\mathbf{x}, {}^*g_2\mathbf{x}) + Q({}^*g_2\mathbf{x}, {}^*g_3\mathbf{y}) + \frac{1}{2}Q({}^*g_3\mathbf{y}, {}^*g_4\mathbf{y})\right) d\mu(\mathbf{y}),\end{aligned} \tag{2.1.9}$$

where ${}^*g$ is defined by $Q(g\mathbf{x},\mathbf{y}) = Q(\mathbf{x}, {}^*g\mathbf{y})$. The Haar measure $d\mu(\mathbf{y})$ on $X_k/\ker(g_3)$ is the unique one that makes this operator unitary with respect to the L^2-norm on $\mathscr{S}(X_k)$. We can then extend the local Weil representation to a global Weil representation $\mathscr{S}(X_{\mathbb{A}_L})$ of $\mathrm{Sp}(W_{\mathbb{A}_L})$.

The operator (2.1.9) only defines a projective representation in the sense that $\omega(g_1g_2) = c(g_1,g_2)\omega(g_1)\omega(g_2)$ for some cocycle $c(g_1,g_2) \in \mathbb{C}^\times$ of norm 1; see [Rao93]. However, on some subgroups the cocycle becomes trivial and we have a true representation of those subgroups.

Dual pairs and theta kernels. A *dual pair* (G,H) is a pair of subgroups G and H of $\mathrm{Sp}(W_L)$ that centralize each other in $\mathrm{Sp}(W_L)$. This allows us to view $G \times H \subset \mathrm{Sp}(W_L)$ as a subgroup and to restrict the Weil representation to this product. If $\varphi \in \mathscr{S}(X_{\mathbb{A}_L})$ is a Schwartz function we define for $(g,h) \in G(\mathbb{A}_L) \times H(\mathbb{A}_L)$ the theta kernel

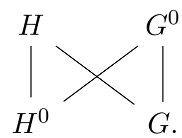
$$\Theta(g,h,\varphi) := \sum_{\mathbf{x}\in X_L} \omega(g,h)\varphi(\mathbf{x}).$$

Let F in $C^\infty(H(L)\backslash H(\mathbb{A}_L))$ be an automorphic form on H. We consider the integral

$$I(g,\varphi,F) := \int_{H(L)\backslash H(\mathbb{A}_L)} \Theta(g,h,\varphi)F(h)dh,$$

which in general does not converge. If it does converge, then it defines a function $I(\varphi,F)$ on G, which is an automorphic form on G by the work of Weil [Wei64]. We call it the *theta lift of* F.

The seesaw principle A *seesaw pair* is a pair of dual pairs (G^0,H^0) and (G,H) such that $H^0\subset H$ and $G\subset G^0$. Such a pair is represented by a diagram



Let $\Theta^0(g^0,h^0,\varphi)$ be the theta kernel associated to the dual pair (G^0,H^0). Both subgroups $G\times H$ and $G^0\times H^0$ can be restricted to the common subgroup $G\times H^0$. Hence we have

$$\Theta^0(g,h^0,\varphi) = \Theta(g,h^0,\varphi).$$

Suppose we have an automorphic form F^0 on H^0 and consider the theta lift

$$I^0(g^0,\varphi,F^0) := \int_{H^0(L)\backslash H^0(\mathbb{A}_L)} \Theta(g^0,h^0,\varphi)F(h^0)dh^0,$$

which is an automorphic form on G^0. The restriction $I^0(\varphi,F^0)\big|_G$ of $I^0(g^0,\varphi,F^0)$ to G is an automorphic form. The restriction $F\big|_{H^0}$ of F to H^0 is an automorphic form on H^0. We have the equality

$$I^0(\varphi,F\big|_{H^0})\big|_G = I(\varphi,F),$$

and we call it the *see-saw principle*.

Examples of dual pairs. We will be interested in a seesaw that is obtained by restriction of scalars. It is the example (2.19) given by Kudla in [Kud84] and will be explained in Section 4.3.1

There are two types of dual pairs that will be involved in Chapter 4. Let k be one of the completions of L and $f_1,\ldots,f_{2m}$ a k-basis of X_k. This basis identifies W_k with k^{4m}, where the symplectic form

$$\begin{pmatrix} 0 & A(Q) \\ -A(Q) & 0 \end{pmatrix}$$

where $A(Q) \in M_{2m}(k)$ is the symmetric matrix $(A(Q))_{ij} = Q(f_i, f_j)$. In this basis the symplectic group is given by

$$\mathrm{Sp}(W_k) = \left\{ g \in \mathrm{GL}_{4m}(k) \,\middle|\, {}^{\mathrm{t}}g \begin{pmatrix} & A(Q) \\ -A(Q) & \end{pmatrix} g = \begin{pmatrix} & A(Q) \\ -A(Q) & \end{pmatrix} \right\}.$$

1. We view X as k^{2m} with the quadratic form $A(Q)$. We can embedd $\mathrm{GL}_{2m}(k) \subset \mathrm{Sp}(W_k)$ as

$$\begin{aligned} \mathrm{GL}_{2m}(k) &\hookrightarrow \mathrm{Sp}(W_k) \\ g &\longmapsto \begin{pmatrix} g & \\ & {}^{*}g^{-1} \end{pmatrix}, \end{aligned}$$

   where

$$^{*}g \coloneqq A(Q)^{-1}\, {}^{\mathrm{t}}g A(Q)$$

   is as above. The centralizer of $\mathrm{GL}_{2m}(k)$ is the center $\mathrm{GL}_1(k) \subset \mathrm{GL}_{2m}(k)$, and we obtain a dual pair $\mathrm{GL}_{2m} \times \mathrm{GL}_1$ which we call the *linear pair*. The operator 2.1.9 defines a Weil representation of $\mathrm{GL}_{2m}(k) \times \mathrm{GL}_1(k)$ on $\mathscr{S}(X_k)$ by

$$\omega_l(g,t)\varphi(\mathbf{x}) = |\det(gt)|^{\frac{1}{2}} \varphi\left({}^{*}gt\mathbf{x}\right).$$

2. We can restrict the embedding to the orthogonal group of X_k

$$\begin{aligned} \mathrm{SO}(X_k) &\hookrightarrow \mathrm{Sp}(W_k) \\ h &\longmapsto \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \end{aligned}$$

since ${}^*h^{-1} = h$. Its centralizer is isomorphic to $\mathrm{SL}_2(k)$, embedded as follows

$$\begin{aligned} \mathrm{SL}_2(k) &\longhookrightarrow \mathrm{Sp}(W_k) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} a & & b & \\ & a & & b \\ c & & d & \\ & c & & d \end{pmatrix}. \end{aligned}$$

Hence the pair $\mathrm{SO}(X_k) \times \mathrm{SL}_2(k)$ is the second example of dual pair, that we call the *orthosymplectic pair*. The restriction of the operators defined in (2.1.9) yields the Weil representation on $\mathscr{S}(X_k)$ for the pair $\mathrm{SL}_2(k) \times \mathrm{SO}(X_k)$:

$$\omega_{os}(g,h)\varphi(\mathbf{x}) = \Big(\omega_{os}(g,1)\varphi\Big)(h^{-1}\mathbf{x}),$$

$$\omega_{os}\left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, 1\right)\varphi(\mathbf{x}) \coloneqq \chi_k\left(\frac{bQ(\mathbf{x},\mathbf{x})}{2}\right)\varphi(\mathbf{x}), \tag{2.1.10}$$

$$\omega_{os}\left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, 1\right)\varphi(\mathbf{x}) \coloneqq |a|^m\varphi(a\mathbf{x}), \tag{2.1.11}$$

$$\omega_{os}(S,1)\varphi(\mathbf{x}) \coloneqq \int_{X_k}\varphi(\mathbf{y})\chi_k(-Q(\mathbf{x},\mathbf{y}))\,d\mu(\mathbf{y}),$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The assumption that X_L is even dimensional implies that the restriction of the projective representation (2.1.9) defines a true representation on $\mathrm{SL}_2(k) \times \mathrm{SO}(X_k)$. For more on the Weil representation and the theta correspondence we refer to [Wei64], [Rao93], [LV80] and [MVW87].

2.2 Kudla-Millson form and special cycles

Let us now summarize the work of Kudla and Millson in the case of the dual pair $\mathrm{SL}_2 \times \mathrm{SO}_V$. For more details we refer to [KM86], [KM87], [KM90] and [Kud97].

2.2.1 The symmetric space $\mathbb{D}^+$

Let $(X_{\mathbb{Q}}, Q)$ be a rational quadratic space, let (p,q) be the signature of $X_{\mathbb{R}}$ and $H = \mathrm{SO}(X_{\mathbb{Q}})$ the orthogonal group. Let $\mathbf{e}_1, \dots, \mathbf{e}_p, \mathbf{e}_{p+1}, \cdots, \mathbf{e}_{p+q}$ be an orthogonal basis of $X_{\mathbb{R}}$, where the first p vectors have norm 1 and the last q have norm -1. The Lie group $H(\mathbb{R})$ can be identified with $\mathrm{SO}(p,q)$ and has two connected components. We denote by $H(\mathbb{R})^+$ the connected component of the identity. Let $\mathbb{D}$ be the space of *oriented* negative q-planes

$$\mathbb{D} := \left\{ z \subset X_{\mathbb{R}} \mid z \,\text{oriented}, \ \dim(z) = q, \quad Q\big|_z < 0 \right\};$$

it is a pq-dimensional Riemannian manifold. This space has two connected components $\mathbb{D} = \mathbb{D}^+ \sqcup \mathbb{D}^-$. For an oriented subspace z let $o(z) \in \wedge^N z$ be its orientation.

Let z_0 be the q-plane spanned by $\mathbf{e}_{p+1}, \cdots, \mathbf{e}_{p+q}$ with the orientation $o(z_0) = \mathbf{e}_{p+1} \wedge \cdots \wedge \mathbf{e}_{p+q}$. Let $\mathbb{D}^+$ be the connected component containing the oriented plane z_0. The group $H(\mathbb{R})^+$ acts transitively on $\mathbb{D}^+$ by sending z_0 to hz_0. Thus we can identify $\mathbb{D}^+$ with $H(\mathbb{R})^+/K_\infty(z_0)$ where $K_\infty(z_0)$ is the stabilizer of z_0 in $H(\mathbb{R})^+$ and can be identified with $\mathrm{SO}(p) \times \mathrm{SO}(q)$.

It has the structure of a Riemannian manifold as follows. Let z be a fixed negative plane in $\mathbb{D}^+$. Any plane z' in $\mathbb{D}^+$ can be viewed as the graph of a linear map from z to $z^\perp$. More precisely this map is defined as follows: since z and z' are both negative planes we have $z' \cap z^\perp = 0$ and the orthogonal projection of z' onto z is an isomorphism. For every $\mathbf{u} \in z$ there is a unique $\mathbf{u}' \in z'$ that projects onto $\mathbf{u}$. This defines a homomorphism $\mathbf{u} \longmapsto \mathbf{u} - \mathbf{u}' \in z^\perp$ that depends on z'. Hence we obtain charts $\mathbb{D}^+ \longrightarrow \mathrm{Hom}(z, z^\perp) = z^\vee \otimes z^\perp$ for $\mathbb{D}^+$, where $z^\vee = \mathrm{Hom}(z, \mathbb{R})$ is the dual of z. After differentiating we get an isomorphism

$$T_z\mathbb{D}^+ \longrightarrow z^\vee \otimes z^\perp. \tag{2.2.1}$$

Let $\mathbf{v}^\vee(-) \in z^\vee$ be the dual of $\mathbf{v} \in z$ defined by $\mathbf{v}^\vee(-) := \frac{Q(\mathbf{v},-)}{Q(\mathbf{v},\mathbf{v})}$. We define a Riemannian metric on $\mathbb{D}^+$ by

$$\begin{aligned} q_z \colon z^\vee \otimes z^\perp &\longrightarrow \mathbb{R} \\ \mathbf{u}^\vee \otimes \mathbf{v} &\longmapsto -Q(\mathbf{u})Q(\mathbf{v}). \end{aligned}$$

2.2.2 Symmetric subspaces

Let U be a non-degenerate subspace of $X_{\mathbb{R}}$ of signature (r, s). We define the totally geodesic submanifold

$$\mathbb{D}_U := \left\{ z \in \mathbb{D} \mid z = z \cap U + z \cap U^\perp \right\}$$

and $\mathbb{D}_U^+ = \mathbb{D}^+ \cap \mathbb{D}_U$. If $U = \langle \mathbf{x} \rangle$ is spanned by a single vector we write $\mathbb{D}_{\mathbf{x}}$ instead of $\mathbb{D}_{\langle \mathbf{x} \rangle}$ and it coincides with the submanifolds $\mathbb{D}_{\mathbf{x}}$ defined in the introduction. Suppose that z_0 lies in $\mathbb{D}_U$. Let $H_U(\mathbb{R})$ be the stabilizer of U in $H(\mathbb{R})$ and let $K_U(z_0) := H_U^+(\mathbb{R}) \cap K_\infty(z_0)$. We then have a diffeomorphism

$$\begin{aligned} H_U^+(\mathbb{R})/K_U(z) &\longrightarrow \mathbb{D}_U^+ \subset \mathbb{D}^+ \\ hK_U(z) &\longmapsto hz. \end{aligned}$$

If U is positive definite, then for any negative plane z the intersection $z \cap U$ is empty. Hence

$$\mathbb{D}_U := \left\{ z \in \mathbb{D} \mid z \subset U^\perp \right\}.$$

Similarly if U is negative definite then

$$\mathbb{D}_U := \left\{ z \in \mathbb{D} \mid U \subset z \right\}.$$

Let z be a negative plane in $\mathbb{D}^+$. With respect to the orthogonal splitting $X_\mathbb{R} = z^\perp \oplus z$ the quadratic form splits

$$Q(\mathbf{x},\mathbf{x}) = Q\big|_{z^\perp}(\mathbf{x},\mathbf{x}) + Q\big|_{z}(\mathbf{x},\mathbf{x}).$$

We define the *Siegel majorant at z* to be the positive definite quadratic form

$$Q_z^+(\mathbf{x},\mathbf{x}) := Q\big|_{z^\perp}(\mathbf{x},\mathbf{x}) - Q\big|_{z}(\mathbf{x},\mathbf{x}).$$

2.2.3 Orientations

Let us orient the spaces $\mathbb{D}^+$ and $\mathbb{D}_\mathbf{x}^+$. We fix orientations

$$o(X_\mathbb{R}) := \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p \wedge \mathbf{e}_{p+1} \wedge \cdots \wedge \mathbf{e}_{p+q}, \qquad o(z_0) := \mathbf{e}_{p+1} \wedge \cdots \wedge \mathbf{e}_{p+q}$$

of $X_\mathbb{R}$ and of z_0, let us explain how this yields an orientation on $\mathbb{D}^+$. If $z \in \mathbb{D}^+$ is a plane, we write $z = h_z z_0$ for some $h_z \in H(\mathbb{R})^+$. Let $\mathbf{e}_k(z) = h_z \mathbf{e}_k$ so that $z = \operatorname{span}\big\langle \mathbf{e}_{p+1}(z), \cdots, \mathbf{e}_{p+q}(z) \big\rangle$ and the orientation of z is $\mathbf{e}_{p+1}(z) \wedge \cdots \wedge \mathbf{e}_{p+q}(z)$. We identify $T_z\mathbb{D}^+ \simeq z^\vee \otimes z^\perp$ as in (2.2.1), so that we have to orient $z^\vee \otimes z^\perp$. For an oriented plane z we orient $z^\perp$ by the rule that $o(z^\perp) \wedge o(z) = o(X_\mathbb{R})$. Then we orient $z^\vee$ by $\mathbf{e}_{p+1}^\vee(z) \wedge \cdots \wedge \mathbf{e}_{p+q}^\vee(z)$. Finally, we need to orient the tensor product. If V and W are two vectors spaces with orientations $v_1 \wedge \cdots \wedge v_N$ and $w_1 \wedge \cdots \wedge w_M$ then the we orient the basis $v_i \otimes w_j$ with the lexicographic order from right to left.

Example. *If $o(V) = v_1 \wedge v_2$ and $W = w_1 \wedge w_2$ then $o(V \otimes W) = (v_1 \otimes w_1) \wedge (v_2 \otimes w_1) \wedge (v_1 \otimes w_2) \wedge (v_2 \otimes w_2)$.*

Let us now orient $\mathbb{D}_\mathbf{x}^+$, given a positive vector $\mathbf{x}$. If $z' \in \mathbb{D}_\mathbf{x}^+ \cap \mathcal{V}_z$ then $z' \in \mathbf{x}^\perp$ represents a graph in $\mathbf{x}^\perp$ of a linear map from z to $z^\perp \cap \mathbf{x}^\perp$. As in (2.2.1) we get an isomorphism

$$T_z\mathbb{D}^+_{\mathbf{x}} \longrightarrow z^\vee \otimes (z^\perp \cap \mathbf{x}^\perp).$$

With the Riemannian metric on $\mathbb{D}^+$ the normal bundle $N_z\mathbb{D}^+_{\mathbf{x}}$ is

$$N_z\mathbb{D}^+_{\mathbf{x}} = z^\vee \otimes \mathbf{x} \simeq z^\vee.$$

The space $N_z\mathbb{D}^+_{\mathbf{x}}$ is oriented by $z^\vee$ and we orient $T_z\mathbb{D}^+_{\mathbf{x}}$ by the rule $o(T_z\mathbb{D}^+_{\mathbf{x}}) \wedge o(N_z\mathbb{D}^+_{\mathbf{x}}) = o(T_z\mathbb{D}^+)$.

2.2.4 The Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$

Let

$$\mathfrak{h} := \left\{ \left(\begin{array}{cc} A & x \\ {}^t x & B \end{array} \right) \middle| A \in \mathfrak{so}(z_0^\perp),\ B \in \mathfrak{so}(z_0),\ x \in \mathrm{Hom}(z_0, z_0^\perp) \right\},$$

$$\mathfrak{k} := \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & B \end{array} \right) \middle| A \in \mathfrak{so}(z_0^\perp),\ B \in \mathfrak{so}(z_0) \right\},$$

be the Lie algebras of $H(\mathbb{R})^+$ and K_∞, where $\mathfrak{so}(z_0) = \mathfrak{so}(q)$ is the space of skew-symmetric q by q matrices, and similarly $\mathfrak{so}(z_0^\perp) = \mathfrak{so}(p)$ are the skew symmetric p by p matrices. Hence we have a decomposition

$$\mathfrak{k} = \mathfrak{so}(z_0^\perp) \oplus \mathfrak{so}(z_0),$$

that is orthogonal with respect to the Killing form. The map $\epsilon(X) = -\,{}^{\mathrm{t}}X$ is an involution of the Lie algebra $\mathfrak{h}$. The $+1$-eigenspace is $\mathfrak{k}$, the -1-eigenspace is

$$\mathfrak{p} := \left\{ \left(\begin{array}{cc} 0 & x \\ {}^t x & 0 \end{array} \right) \middle| x \in \mathrm{Hom}(z_0, z_0^\perp) \right\},$$

and we have a decomposition

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$$

which is orthogonal with respect to the Killing form. Since ϵ is a Lie algebra automorphism we have $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ and $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$. We identify the tangent space of $\mathbb{D}^+$ at eK_∞ with $\mathfrak{p} \simeq \mathfrak{h}/\mathfrak{k}$ and the tangent bundle is

$$T\mathbb{D}^+ \simeq H(\mathbb{R})^+ \times_{K_\infty} \mathfrak{p},$$

where K_∞ acts on $\mathfrak{p}$ by the Ad-representation. We have an isomorphism

$$\begin{aligned} T\colon \wedge^2 X_{\mathbb{R}} &\longrightarrow \mathfrak{h} \subset \mathrm{End}(X_{\mathbb{R}}) \\ \mathbf{e}_i \wedge \mathbf{e}_j &\longmapsto T(\mathbf{e}_i \wedge \mathbf{e}_j)\mathbf{e}_k \coloneqq Q(\mathbf{e}_i, \mathbf{e}_k)\mathbf{e}_j - Q(\mathbf{e}_j, \mathbf{e}_k)\mathbf{e}_i. \end{aligned} \tag{2.2.2}$$

A basis of $\mathfrak{h}$ is given by $X_{ij} \coloneqq T(e_i \wedge e_j)$ for $i < j$, and we denote by ω_{ij} the dual basis of $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{R})$. Let E_{ij} be the elementary matrix sending $\mathbf{e}_i$ to $\mathbf{e}_j$ and the other $\mathbf{e}_k$'s to 0. Then $\mathfrak{p}$ is spanned by the matrices

$$X_{\alpha\mu} = E_{\alpha\mu} + E_{\mu\alpha}$$

and $\mathfrak{k}$ is spanned by the matrices

$$\begin{aligned} X_{\alpha\beta} &= E_{\alpha\beta} - E_{\beta\alpha}, \\ X_{\nu\mu} &= -E_{\nu\mu} + E_{\mu\nu}. \end{aligned}$$

2.2.5 Adelic spaces

Let $K \coloneqq K_\infty K_f$ where K_f is an open compact subgroup preserving a Schwartz function $\varphi_f \in \mathscr{S}(X_{\mathbb{A}_f})$ and $K_\infty = K_\infty(z_0)$ as before, the maximal compact subgroup of $H(\mathbb{R})^+$ stabilizing z_0. We define the double coset space

$$M_K \coloneqq H(\mathbb{Q})\backslash H(\mathbb{A})/K \simeq H(\mathbb{Q})\backslash \mathbb{D} \times H(\mathbb{A}_f)/K_f,$$

where the second isomorphism sends $H(\mathbb{Q})(h_\infty, h_f)K \longrightarrow H(\mathbb{Q})(h_\infty z_0, h_f)K$. There exists $h_1, \dots, h_r \in H(\mathbb{A}_f)$ such that

$$H(\mathbb{A}_f) = \bigsqcup_{i=1}^{r} H(\mathbb{Q})^+ h_i K_f.$$

Let $H^{\mathrm{ad}}(\mathbb{R})^+ \coloneqq H(\mathbb{R})^+/Z_H(\mathbb{R})$. Let $\Gamma'_{h_i} \coloneqq H(\mathbb{Q})^+ \cap h_i K_f h_i^{-1}$ and Γ_{h_i} its image in $H^{\mathrm{ad}}(\mathbb{Q})^+ = H^{\mathrm{ad}}(\mathbb{R})^+ \cap H^{\mathrm{ad}}(\mathbb{Q})$. We have a homeomorphism

$$M_K \simeq \bigsqcup_{i=1}^{r} M_{h_i}, \tag{2.2.3}$$

where M_h is the locally symmetric space

$$M_h := \Gamma_h \backslash \mathbb{D}^+.$$

If Γ_h is torsion-free then M_h is an orientable Riemannian manifold of dimension pq. If Γ_h has torsion, then M_h has the structure of an orbifold and there is a finite index torsion-free subgroup $\widetilde{\Gamma}_h \subset \Gamma_h$.

The map 2.2.3 goes as follows. Let $H(\mathbb{Q})(z, h_f)K$ be a double coset. Let $\delta_z = 1$ if $z \in \mathbb{D}^+$ and δ_z is any element $H(\mathbb{Q}) - H(\mathbb{Q})^+$ if $z \in \mathbb{D}^-$. The element δ_z permutes $\mathbb{D}^+$ and $\mathbb{D}^-$, hence

$$H(\mathbb{Q})(z, h_f)K = H(\mathbb{Q})(\delta_z z, \delta_z h_f)K$$

with $\delta_z z \in \mathbb{D}^+$. There is an $i \in \{1, \dots, r\}$ we write $\delta_z h_f = h^{-1} h_i k_f$ for some $k_f \in K_f$ and $h \in H(\mathbb{Q})^+$. Then this coset is mapped to the point $\Gamma_{h_i} h \delta_z z$.

2.2.6 The Kudla-Millson form

Let us write $\Gamma = \Gamma_{h_i}$ for any of the h_i's as in (2.2.3). In a series of papers, Kudla and Millson [KM86; KM87; KM90] defined a form

$$\varphi_{KM} \in \left[\mathscr{S}(X_{\mathbb{R}}) \otimes \Omega^q(\mathbb{D}^+)\right]^{H(\mathbb{R})^+}.$$

This form satisfies

1. $\varphi_{KM}(\mathbf{x})$ is closed for any $\mathbf{x} \in X_{\mathbb{R}}$,
2. φ_{KM} is $H(\mathbb{R})^+$-invariant in the sense that

$$h^* \varphi_{KM}(\mathbf{x}) = \varphi_{KM}(h^{-1}\mathbf{x}) = \Big(\omega_{os}(h)\varphi_{KM}\Big)(\mathbf{x}).$$

   In particular it is $\Gamma_{\mathbf{x}}$-invariant, where $\Gamma_{\mathbf{x}}$ is the stabilizer of $\mathbf{x}$ in Γ, and we can view $\varphi_{KM}(\mathbf{x})$ as a form on $\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+$.

3. For positive vectors $\mathbf{x}$ and $\omega \in \Omega_c^{pq-q}(\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+)$ the form $\varphi_{KM}(\mathbf{x})$ satisfies

$$\int_{\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+} \varphi_{KM}(\mathbf{x}) \wedge \omega = 2^{-\frac{q}{2}} e^{-\pi Q(\mathbf{x},\mathbf{x})} \int_{\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+_{\mathbf{x}}} \omega,$$

   where $\Omega_c^{\bullet}$ denotes the space of compactly supported forms.

The last property means that the form

$$\varphi^0(\mathbf{x}) := 2^{\frac{q}{2}} e^{\pi Q(\mathbf{x},\mathbf{x})} \varphi_{KM}(\mathbf{x}) \in \Omega^q(\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+)$$

is a Poincaré dual of $\Gamma_{\mathbf{x}}\backslash\mathbb{D}^+_{\mathbf{x}}$ in $\Gamma_{\mathbf{x}}\backslash\mathbb{D}^+$.

Remark 2.2.1. The Kudla-Millson form defined here is the original Kudla-Millson form multiplied by $2^{\frac{q}{2}}$.

In the work of Kudla and Millson the form is constructed as follows. The tangent plane of $\mathbb{D}^+$ is $T_{eK_\infty}\mathbb{D}^+ \simeq \mathfrak{p}$, hence the cotangent bundle is

$$(T\mathbb{D}^+)^* = H(\mathbb{R})^+ \times_{K_\infty} \mathfrak{p}^*,$$

where K_∞ acts on $\mathfrak{p}$ by the dual Ad-representation. The basis $\mathbf{e}_1, \dots, \mathbf{e}_{p+q}$ identifies $X_\mathbb{R}$ with $\mathbb{R}^{p+q}$. With respect to this basis the Siegel majorant at z_0 is given by

$$Q^+_{z_0}(\mathbf{x},\mathbf{x}) = \sum_{i=1}^{p+q} x_i^2.$$

Recall that $H(\mathbb{R})^+$ acts on $\mathscr{S}(\mathbb{R}^{p+q})$ by $(h \cdot f)(\mathbf{x}) = f(h^{-1}\mathbf{x})$. We have an isomorphism

$$\begin{aligned} \left[\Omega^q(\mathbb{D}^+) \otimes \mathscr{S}(\mathbb{R}^{p+q})\right]^{H(\mathbb{R})^+} &\longrightarrow \left[\bigwedge\nolimits^q \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q})\right]^{K_\infty} \\ \varphi &\longrightarrow \varphi_e \end{aligned} \tag{2.2.4}$$

by evaluating φ at the basepoint eK_∞, corresponding to z_0 in $\mathbb{D}^+$. We define the *Howe operator*

$$D\colon \bigwedge\nolimits^\bullet \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q}) \longrightarrow \bigwedge\nolimits^{\bullet+q} \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q})$$

by

$$D \coloneqq \frac{1}{2^q} \prod_{\mu=p+1}^{p+q} \sum_{\alpha=1}^{p} A_{\alpha\mu} \otimes \left(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\right)$$

where $A_{\alpha\mu}$ is left multiplication by $\omega_{\alpha\mu}$, the dual of $X_{\alpha\mu}$. The Kudla-Millson form is defined by applying D to the Gaussian:

$$\varphi_{KM}(\mathbf{x})_e \coloneqq D \exp\left(-\pi Q^+_{z_0}(\mathbf{x},\mathbf{x})\right) \in \bigwedge\nolimits^q \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q}). \tag{2.2.5}$$

In Chapter 3 we will see that the Kudla-Millson form can also be computed via Chern-Weil theory and without the Howe operators.

2.2.7 Homology and cohomology of M_K

We can identify $\Omega^\bullet(M_h)$ with $\Omega^\bullet(\mathbb{D}^+)^{\Gamma_h}$, where the latter is the space of Γ_h-invariant forms on $\mathbb{D}^+$. We use the isomorphism (2.2.3) to define the space of differential forms on M_K:

$$\Omega^\bullet(M_K) \coloneqq \bigoplus_{i=1}^{r} \Omega^\bullet(M_{h_i})$$

and the cohomology of M_K by

$$H^\bullet(M_K;\mathbb{R}) \coloneqq \bigoplus_{i=1}^{r} H^\bullet(M_{h_i};\mathbb{R}).$$

Similarly we define the homology of M_K by

$$H_\bullet(M_K;\mathbb{R}) \coloneqq \bigoplus_{i=1}^{r} H_\bullet(M_{h_i};\mathbb{R}).$$

Let $C^\infty(H(\mathbb{A}_f))$ be the space of smooth (*i.e.* locally constant) functions. We can also see $\Omega^\bullet(M_K)$ as

$$\begin{aligned}\Omega^\bullet(M_K) &\simeq \left[\Omega^\bullet(\mathbb{D}^+)\otimes_{\mathbb{Q}} C^\infty(H(\mathbb{A}_f))\right]^{H(\mathbb{Q})\times K_f} \\ &\simeq \left[C^\infty(H(\mathbb{Q})\backslash H(\mathbb{A}))\otimes_{\mathbb{R}} \bigwedge^\bullet \mathfrak{p}^*\right]^K, \end{aligned} \tag{2.2.6}$$

where $\mathfrak{p}^* = \mathrm{Hom}(\mathfrak{p},\mathbb{R})$. The first isomorphism is

$$\omega\otimes f \longmapsto \sum_{i=1}^{r} f(h_i)\omega \in \bigoplus_{i=1}^{r} \Omega^\bullet(\mathbb{D}^+)^{\Gamma_{h_i}}$$

and the second ones comes from the isomorphism

$$\Omega^\bullet(\mathbb{D}^+) \simeq \left[C^\infty(H(\mathbb{R})^+)\otimes_{\mathbb{R}} \bigwedge^\bullet \mathfrak{p}^*\right]^{K_\infty}.$$

2.2.8 Special cycles

For $h_i \in H(\mathbb{A}_f)$ and $\mathbf{x}\in X_{\mathbb{Q}}$ we define the connected cycles $C_{\mathbf{x}}(h_i)\subset M_{h_i}$ to be the image of the composition

$$\Gamma_{h_i,\mathbf{x}}\backslash\mathbb{D}^+_{\mathbf{x}} \hookrightarrow \Gamma_{h_i,\mathbf{x}}\backslash\mathbb{D}^+ \longrightarrow M_{h_i}$$

where $\Gamma_{h_i,\mathbf{x}} \coloneqq H_{\mathbf{x}}(\mathbb{Q})^+ \cap h_i K h_i^{-1}$. Note that $\Gamma_{h_i,\mathbf{x}}$ does not contain -1 so its image in $H^{\mathrm{ad}}(\mathbb{Q})^+$ is $\Gamma_{h_i,\mathbf{x}}$. For $n \in \mathbb{Q}_{>0}$ we define the *weighted cycles*

$$C_n(\varphi_f, h_i) \coloneqq \sum_{\substack{\mathbf{x}\in\Gamma'_{h_i}\backslash X_{\mathbb{Q}} \\ Q(\mathbf{x},\mathbf{x})=2n}} \varphi_f(h_i^{-1}\mathbf{x}) C_{\mathbf{x}}(h_i) \in \mathcal{Z}_{pq-q}(\overline{M_{h_i}}, \partial\overline{M_{h_i}}; \mathbb{R})$$

$$C_n(\varphi_f) \coloneqq \sum_{i=1}^{r} C_n(\varphi_f, h_i) \in \mathcal{Z}_{pq-q}(\overline{M_K}, \partial\overline{M_K}; \mathbb{R}).$$

We see this cycle as a formal linear combination of the cycles $C_{\mathbf{x}}(h_i)$ of codimension q, one in each connected component M_{h_i} of M_K.

2.2.9 The Kudla-Millson theta series

Let us now suppose that $p+q$ is even. In that case the Weil representation for $\mathrm{SL}_2(\mathbb{Q}) \times \mathrm{SO}(X_{\mathbb{Q}})$ is a true (*i.e.* not projective) representation. We define the theta series for $g \in \mathrm{SL}_2(\mathbb{A})$ and $h_f \in H(\mathbb{A}_f)$

$$\begin{aligned}\Theta_{os}(g, h_f, \varphi_{KM} \otimes \varphi_f) &\coloneqq \sum_{\mathbf{x}\in X_{\mathbb{Q}}} \Big(\omega_{os}(g, h_f)\varphi_{KM} \otimes \varphi_f\Big)(\mathbf{x}) \\ &= \sum_{\mathbf{x}\in X_{\mathbb{Q}}} \Big(\omega_{os}(g)\varphi_{KM}\Big)(\mathbf{x})\varphi_f(h_f^{-1}\mathbf{x}) \in \Omega^q(\mathbb{D}^+).\end{aligned}$$

The form $\Theta_{os}(g, h_i, \varphi_f)$ is Γ_{h_i}-invariant, hence defines a form on M_{h_i} and we obtain a form

$$\Theta_{os}(g, \varphi_{KM} \otimes \varphi_f) \coloneqq \sum_{i=1}^{r} \Theta_{os}(g, h_i, \varphi_{KM} \otimes \varphi_f) \in \Omega^q(M_K).$$

For $\tau = u + iv$ a point in $\mathbb{H}$ we define the classical Kudla-Millson theta series

$$\Theta_{KM}(\tau, h_f, \varphi_f) \coloneqq v^{-\frac{p+q}{4}} \Theta_{os}(g_\tau, h_f, \varphi_{KM} \otimes \varphi_f) \tag{2.2.7}$$

where $g_\tau = \begin{pmatrix} \sqrt{v} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ maps i to τ.

Remark 2.2.2. We have $\omega_{os}(k_\theta)\varphi_{KM} = e^{i\theta\frac{p+q}{2}}$ for

$$k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in \mathrm{SO}(2).$$

Since ω_{os} is a representation, one can check that (2.2.7) does not depend on the choice of the matrix g_τ sending i to τ.

By summing over the different connected components we then also get a form

$$\Theta_{KM}(\tau,\varphi_f) \coloneqq \sum_{i=1}^{r} \Theta_{KM}(\tau,h_i,\varphi_f) \in \Omega^q(M_K).$$

Moreover this form is closed, so that is represents a cohomology class

$$[\Theta_{KM}(\tau,\varphi_f)] \in H^q(M_K;\mathbb{R}).$$

Lemma 2.2.1. *We can rewrite the theta series as*

$$\Theta_{KM}(\tau,\varphi_f) = \Theta_0(v,\varphi_f) + \sum_{n\in\mathbb{Q}^\times} \Theta_n(v,\varphi_f)e^{2i\pi n\tau}$$

where

$$\Theta_n(v,\varphi_f) = \sum_{i=1}^{r} \sum_{\substack{\mathbf{x}\in X_{\mathbb{Q}} \\ Q(\mathbf{x},\mathbf{x})=2n}} \varphi_f(h_i^{-1}\mathbf{x})\varphi^0(\sqrt{v}\mathbf{x}).$$

Proof. It follows from the formulas (2.1.10)-(2.1.11) for the Weil representation that

$$v^{-\frac{N}{2}}\omega_{os}(g_\tau)\varphi_{KM}(\mathbf{x}) = \varphi^0(\sqrt{v}\mathbf{x})e^{i\pi\tau Q(\mathbf{x},\mathbf{x})}$$

where $\varphi^0(\mathbf{x}) = e^{\pi Q(\mathbf{x},\mathbf{x})}\varphi_{KM}(\mathbf{x})$. Thus

$$\Theta_{KM}(\tau,\varphi_f) = \sum_{i=1}^{r}\sum_{\mathbf{x}\in X_{\mathbb{Q}}} \varphi_f(h_i^{-1}\mathbf{x})\varphi^0(\sqrt{v}\mathbf{x})e^{i\pi\tau Q(\mathbf{x},\mathbf{x})}. \tag{2.2.8}$$

Grouping together the contributions coming from vectors of same length we can rewrite (2.2.8) as

$$\begin{aligned}\Theta_{KM}(\tau,\varphi_f) &= \sum_{n\in\mathbb{Q}} \left(\sum_{i=1}^{r} \sum_{\substack{\mathbf{x}\in X_{\mathbb{Q}} \\ Q(\mathbf{x},\mathbf{x})=2n}} \varphi_f(h_i^{-1}\mathbf{x})\varphi^0(\sqrt{v}\mathbf{x}) \right) e^{2i\pi n\tau} \\ &= \sum_{n\in\mathbb{Q}} \Theta_n(v,\varphi_f)e^{2i\pi n\tau}.\end{aligned}$$

□

Theorem 2.2.2 (Kudla-Millson)**.** *Let $C = \sum C_i \in \mathcal{Z}_q(M_K; \mathbb{R})$ be a cycle, then*

$$\int_C \Theta_{KM}(\tau, \varphi_f) = \sum_{i=1}^{r} \int_{C_i} \Theta_{KM}(\tau, h_i, \varphi_f)$$

is a modular form of weight $\frac{p+q}{2}$. Moreover, it admits the Fourier decomposition

$$\int_C \Theta_{KM}(\tau, \varphi_f) = \int_C \Theta_0(v, \varphi_f) + \sum_{n \in \mathbb{Q}_{>0}} \int_C \Theta_n(v, \varphi_f) e^{2i\pi n\tau}$$

and for $n > 0$ we have

$$\int_C \Theta_n(v, \varphi_f) = \kappa \langle C_n(\varphi_f), C \rangle.$$

In particular, they show the holomorphicity (in τ) of the resulting function and the vanishing of the Fourier coefficients for $n < 0$. In Chapter 4 we will consider a quadratic space of signature (N, N) and the integral of $\Theta_{KM}(\tau, \varphi_f)$ over a relative cycle

$$C \otimes \psi \in \mathcal{Z}_N(\overline{M_K}, \partial \overline{M_K}; \mathbb{R}),$$

which is twisted by a totally odd unitary Hecke character ψ. In particular since $C \otimes \psi$ will be non-compact the result of Kudla-Millson does not apply immediately. Using a seesaw argument, we will show that

$$\int_{C \otimes \psi} \Theta_{KM}(\tau, \varphi_f)$$

equals the diagonal restriction of an Eisenstein series, which will prove in particular that it is holomorphic. Although the vanishing of the negative Fourier coefficients also follows from this equality, we show it directly in Proposition 4.3.6. Moreover, we will show in Proposition 4.3.10 that the Fourier coefficients can be interpreted as intersection numbers. The proof of the proposition, and the Remark 4.3.2 before it also explains the appearance of the factor κ here.

Remark 2.2.3. Recall that if C, C' are immersed submanifolds, then

$$\int_C \mathrm{PD}(C') = \int_{M_K} \mathrm{PD}(C') \wedge \mathrm{PD}(C) = \langle C', C \rangle, \tag{2.2.9}$$

where the right hand side is the topological intersection numbers. This works when at least one of the two cycles is compact. If the two cycles C, C' are non-compact and intersect infinitely many times, then the integrals (2.2.9) do not converge and the intersection is not well defined. On the other hand, if C and C' intersect finitely many times in M_K then $\langle C, C' \rangle$ is well-defined and the integrals converge. However it does not mean that the equality (2.2.9) holds. Indeed, to make sense of (2.2.9) for non-compact cycles one would need to study the extension of the forms to the boundary of a compactification as well as the intersections of the cycles in that boundary, as it is done in [FM14] for example.

The special cycle $C_n(\varphi_f)$ is an immersed submanifold of codimension q, and for $n > 0$ the form $\kappa^{-1}\Theta_n(v, \varphi_f)$ represents the Poincaré dual of $C_n(\varphi_f)$ in $\Omega^q(M_{h_i})$. If C is compact we get

$$\int_C \Theta_n(v, \varphi_f) = \kappa \langle C_n(\varphi_f), C \rangle. \tag{2.2.10}$$

In our case we will replace C by a non-compact cycle $C \otimes \psi$ that intersects $C_n(\varphi_f)$ in a compact set. We will show in Proposition 4.3.10 that (2.2.10) also holds for $C \otimes \psi$.

Chapter 3

The Kudla-Millson form and the Mathai-Quillen formalism

In this chapter we show how to recover the Kudla-Millson form through a construction of Mathai and Quillen [MQ86]. The main result is Theorem 3.2.5 (Theorem A in the introduction). We recall the setting: let $(X_\mathbb{Q}, Q)$ be an arbitrary rational quadratic space such that $X_\mathbb{R}$ has signature (p,q), and $H = \mathrm{SO}(Q)$ be the orthogonal group of $X_\mathbb{Q}$. Let $\mathcal{L} \subset X_\mathbb{Q}$ be a lattice[1] and $\Gamma \subset H(\mathbb{Q})^+$ a discrete subgroup preserving $\mathcal{L}$. For a vector $\mathbf{x}$ in $\mathcal{L}$ let $\Gamma_\mathbf{x}$ be the stabilizer of $\mathbf{x}$ in Γ. The space $\mathbb{D}^+$ of oriented negative q planes in $X_\mathbb{R}$ has a tautological bundle E, whose fiber over $z \in \mathbb{D}^+$ is the negative plane z itself. This bundle has a natural metric $-Q\big|_z$ on the fiber. After choosing

[1] For example if we start with a Schwartz function $\varphi_f \in \mathscr{S}(X(\mathbb{A}_f))$ that is the characteristic function of an adelic lattice, then we could take $\mathcal{L} = \mathcal{L}_{h_i}(\varphi_f)$ and $\Gamma = \Gamma_{h_i}$ for some $h_i \in H(\mathbb{A}_f)$ as in (1.0.2).

a connection ∇ compatible with the metric, the Mathai-Quillen formalism produces a Thom form $U_{MQ} \in \Omega^q(E)$. We will show that after pulling back by a section $s_{\mathbf{x}} \colon \mathbb{D}^+ \longrightarrow E$, we have

$$\varphi_{KM}(\mathbf{x}) = 2^{-\frac{q}{2}} e^{-\pi Q(\mathbf{x},\mathbf{x})} s_{\mathbf{x}}^* U_{MQ} \in \Omega^q(\mathbb{D}^+). \tag{3.0.1}$$

Remark 3.0.1. Since $\mathbb{D}^+$ and E are contractible, the Thom isomorphism

$$H^{q+i}_{rd}(E) \longrightarrow H^i(\mathbb{D}^+)$$

described in Subsection 2.1.3 is trivial in this case. Thus, the forms U_{MQ} and $\varphi_{KM}(\mathbf{x})$ are both 0 in cohomology, and the equality (3.0.1) becomes trivial in cohomology as well. As we will show in Subsection 3.2.1, the form U_{MQ} is Γ-invariant. In particular it is $\Gamma_{\mathbf{x}}$-invariant or every vector $\mathbf{x}$. Hence U_{MQ} can also be seen as a Thom form of the bundle $\Gamma_{\mathbf{x}}\backslash E \longrightarrow \Gamma_{\mathbf{x}}\backslash \mathbb{D}^+$, for which the Thom isomorphism is non-trivial. Since the section $s_{\mathbf{x}}$ is $\Gamma_{\mathbf{x}}$-invariant we can also view it as a section of the bundle $\Gamma_{\mathbf{x}}\backslash E$. The pullback $s_{\mathbf{x}} U_{MQ}$ defines a form on $\Gamma_{\mathbf{x}}\backslash \mathbb{D}^+$ and Equation (3.0.1) becomes non-trivial in the cohomology group $H^q(\Gamma_{\mathbf{x}}\backslash \mathbb{D}^+)$.

3.1 Bundles and the Mathai-Quillen formalism

We begin by collecting a few facts about bundles and explain the construction of the Mathai-Quillen form in Subsection 3.1.5. For the construction of the Mathai-Quillen form we will also follow [BGV03], and for more on principal bundles and connections we refer to [KN96].

Let $\mathbf{e}_1, \dots, \mathbf{e}_p, \mathbf{e}_{p+1}, \cdots, \mathbf{e}_{p+q}$ be an orthogonal basis of $X_{\mathbb{R}}$ such that

$$\begin{aligned} Q(\mathbf{e}_\alpha, \mathbf{e}_\alpha) &= 1 \quad \text{for } 1 \leq \alpha \leq p, \\ Q(\mathbf{e}_\mu, \mathbf{e}_\mu) &= -1 \quad \text{for } p+1 \leq \mu \leq p+q. \end{aligned} \tag{3.1.1}$$

Note that we will always use letters α and β for indices between 1 and p, and letters μ and ν for indices between $p+1$ and $p+q$. Let z_0 be the negative q-plane spanned by the vectors $\mathbf{e}_{p+1}, \cdots, \mathbf{e}_{p+q}$ and $K_\infty = \mathrm{SO}(p) \times \mathrm{SO}(q)$ be the maximal compact connected subgroup of $H(\mathbb{R})^+$ stabilizing z_0. Let $\mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of $H(\mathbb{R})^+$ and K_∞.

3.1.1 K_∞-principal bundles

Let P be a smooth principal K_∞-bundle where

$$\begin{aligned} R \colon K_\infty \times P &\longrightarrow P \\ (k,p) &\longmapsto R_k(p) \end{aligned}$$

is the smooth right action of K_∞ on P and $\pi\colon P \longrightarrow P/K_\infty$ the projection map. For a fixed p in P consider the map

$$\begin{aligned} R_p\colon K_\infty &\longrightarrow P \\ k &\longmapsto R_k(p). \end{aligned}$$

Let V_pP be the image of the derivative at the identity

$$d_eR_p\colon \mathfrak{k} \longrightarrow T_pP,$$

which is injective. It coincides with the kernel of the differential $d_p\pi$. A vector in V_pP is called a *vertical vector*. Using this map we can view a vector X in $\mathfrak{k}$ as a vertical vector field on P. The space P can a priori be arbitrary, but in our case we will consider either

1. P is $H(\mathbb{R})^+$ and R_k the natural right action sending g to gk. Then P/K_∞ can be identified with $\mathbb{D}^+$,

2. P is $H(\mathbb{R})^+ \times z_0$ and the action R_k maps (g,w) to $(gk, k^{-1}w)$. In this case P/K_∞ can be identified with $H(\mathbb{R})^+ \times_K z_0$. Is the vector bundle associated to the principal bundle $H(\mathbb{R})^+$ as defined below.

A *principal K_∞-connection* on P is a 1-form θ_P in $\Omega^1(P,\mathfrak{k})$ such that

- $\iota_X\theta_P = X \quad$ for any X in $\mathfrak{k}$,
- $R_k^*\theta_P = Ad(k^{-1})\theta_P \quad$ for any k in K_∞,

where ι_X is the interior product

$$\begin{aligned} \iota_X\colon \Omega^k(P) &\longrightarrow \Omega^k(P) \\ \omega &\longmapsto (\iota_X\omega)(X_1,\dots,X_{p-1}) \coloneqq \omega(X,X_1,\dots,X_{p-1}). \end{aligned}$$

and we view X as a vector field on P. Geometrically these conditions imply that the kernel of θ_P defines a horizontal subspace of TP that we denote by HP. It is a complement to the vertical subspace *i.e.* we get a splitting of T_pP as $V_pP \oplus H_pP$.

Let $\mathfrak{g}$ be the Lie algebra of $H(\mathbb{R})^+$ and let p be the orthogonal projection from $\mathfrak{g}$ on $\mathfrak{k}$. After identifying $\mathfrak{g}^*$ with the space $\Omega^1(H(\mathbb{R})^+)^{H(\mathbb{R})^+}$ of $H(\mathbb{R})^+$-invariant forms we define a natural 1-form called the *Maurer-Cartan form*

$$\sum_{1\leq i<j\leq p+q} \omega_{ij} \otimes X_{ij} \in \Omega^1(H(\mathbb{R})^+) \otimes \mathfrak{g},$$

where X_{ij} is the basis of $\mathfrak{g}$ defined earlier and ω_{ij} its dual basis of $\mathfrak{g}^*$. After projection onto $\mathfrak{k}$ we get a form

$$\theta \coloneqq p\left(\sum_{1\leq i<j\leq p+q} \omega_{ij}\otimes X_{ij}\right) \in \Omega^1(H(\mathbb{R})^+)\otimes \mathfrak{k} \tag{3.1.2}$$

where we identify $\Omega^1(H(\mathbb{R})^+,\mathfrak{k})$ with $\Omega^1(H(\mathbb{R})^+)\otimes\mathfrak{k}$. A direct computation shows that it is a principal K-connection on P when P is $H(\mathbb{R})^+$.

If P is $H(\mathbb{R})^+\times z_0$ then the projection

$$\pi\colon H(\mathbb{R})^+\times z_0 \longrightarrow H(\mathbb{R})^+$$

induces a map

$$\pi^*\colon \Omega^1(H(\mathbb{R})^+) \longrightarrow \Omega^1(H(\mathbb{R})^+\times z_0).$$

The form

$$\widetilde{\theta} \coloneqq \pi^*\theta \in \Omega^1(H(\mathbb{R})^+\times z_0)\otimes\mathfrak{k} \tag{3.1.3}$$

is a principal connection on $H(\mathbb{R})^+\times z_0$.

3.1.2 The associated vector bundles

Since z_0 is preserved by K_∞ we have an orthogonal K_∞-representation

$$\begin{aligned}\rho\colon K_\infty &\longrightarrow \mathrm{SO}(z_0)\\ k &\longmapsto \rho(k)w \coloneqq k\big|_{z_0} w,\end{aligned} \tag{3.1.4}$$

where we will usually simply write kw instead of $k\big|_{z_0}w$. We can consider the *associated vector bundle* $P\times_K z_0$ which is the quotient of $P\times z_0$ by K_∞, where K_∞ acts by sending (p,w) to $(R_k(p),\rho(k)^{-1}w)$. Hence an element $[p,w]$ of $P\times_K z_0$ is an equivalence class where the equivalence relation identifies (p,w) with $(R_k(p),\rho(k)^{-1}w)$. This is a vector bundle over P/K_∞ with projection map sending $[p,w]$ to $\pi(p)$. Let $\Omega^i(P/K_\infty, P\times_{K_\infty} z_0)$ be the space of i-forms valued in $P\times_{K_\infty} z_0$, when i is zero it is the space of sections of the associated bundle.

In the two cases of interest to us we define

$$E\coloneqq H(\mathbb{R})^+\times_{K_\infty} z_0, \qquad \widetilde{E}\coloneqq \big(H(\mathbb{R})^+\times z_0\big)\times_{K_\infty} z_0.$$

Note that in both cases P admits a left action of $H(\mathbb{R})^+$ and that the associated vector bundles are

$H(\mathbb{R})^+$-equivariant. Morever it is a Euclidean bundle, equipped with the inner product

$$\langle v, w\rangle := -Q\big|_{z_0}(v, w)$$

on the fiber.

Basic forms. Let $\Omega^i(P, z_0)$ be the space of z_0-valued differential i-forms on P. A differential form α in $\Omega^i(P, z_0)$ is said to be *horizontal* if $\iota_X\alpha$ vanishes for all vertical vector fields X. There is a left action of K_∞ on a differential form α in $\Omega^i(P, z_0)$ defined by

$$k \cdot \alpha := \rho(k)(R_k^*\alpha),$$

and α is K_∞-invariant if it satisfies $k\cdot\alpha = \alpha$ for any k in K_∞ *i.e.* we have $R_k^*\alpha = \rho(k^{-1})\alpha$. We write $\Omega^i(P, z_0)^{K_\infty}$ for the space of K_∞-invariant z_0-valued forms on P. Finally a form that is horizontal and K_∞-invariant is called a *basic form* and the space of such forms is denoted by $\Omega^i(P, z_0)_{bas}$.

Let $X_1, \ldots, X_N$ be tangent vectors of P/K_∞ at $\pi(p)$ and $\widetilde{X}_i$ a tangent vector of P at p that satisfy $d_p\pi(\widetilde{X}_i) = X_i$. There is a map

$$\begin{aligned} \Omega^i(P, z_0)_{bas} &\longrightarrow \Omega^i(P/K_\infty, P \times_K z_0) \\ \alpha &\longmapsto \omega_\alpha \end{aligned}$$

defined by

$$\omega_\alpha\big|_{\pi(p)}(X_1 \wedge \cdots \wedge X_N) = \alpha\big|_p(\widetilde{X}_1 \wedge \cdots \wedge \widetilde{X}_N).$$

Proposition 3.1.1. *The map is well-defined and yields an isomorphism between* $\Omega^i(P/K_\infty, P\times_{K_\infty} z_0)$ *and* $\Omega^i(P, z_0)_{bas}$. *In particular if* z_0 *is* 1*-dimensional then* $\Omega^i(P/K_\infty)$ *is isomorphic to* $\Omega^i(P)_{bas}$.

Proof. In the case where i is zero the horizontally condition is vacuous and the isomorphism simply identifies $\Omega^0(P/K_\infty, P \times_{K_\infty} z_0)$ with $\Omega^0(P, z_0)^{K_\infty}$. We have a map

$$\begin{aligned} \Omega^0(P, z_0)^{K_\infty} &\longrightarrow \Omega^0(P/K_\infty, P \times_{K_\infty} z_0) \\ f &\longmapsto s_f(\pi(p)) := [p, f(p)], \end{aligned}$$

which is well defined since

$$f(R_k(p)) = \rho(k)^{-1} f(p).$$

Conversely every smooth section s in $\Omega^0(P/K_\infty, P\times_K z_0)$ is given by

$$s(\pi(p)) = [p, f_s(p)]$$

for some smooth function f_s in $\Omega^0(P, z_0)^{K_\infty}$. The map sending s to f_s is inverse to the previous one. The proof is similar for positive i. □

Connections on associated bundles. A *covariant derivative* on the vector bundle $P\times_{K_\infty} z_0$ is a differential operator

$$\nabla_P \colon C^\infty(P/K_\infty, P\times_{K_\infty} z_0) \longrightarrow \Omega^1(P/K_\infty, P\times_{K_\infty} z_0)$$

such that for every f in $C^\infty(P/K_\infty)$ we have

$$\nabla_P(fs) = df\otimes s + f\nabla_P(s).$$

The inner product on $P\times_{K_\infty} z_0$ defines a pairing

$$\begin{aligned}\Omega^i(P/K_\infty, P\times_{K_\infty} z_0)\times\Omega^j(P/K_\infty, P\times_{K_\infty} z_0) &\longrightarrow \Omega^{i+j}(P/K_\infty)\\ (\omega_1\otimes s_1, \omega_2\otimes s_2) &\longmapsto \langle\omega_1\otimes s_1, \omega_2\otimes s_2\rangle\\ &= \omega_1\wedge\omega_2\langle s_1, s_2\rangle,\end{aligned}$$

and we say that the derivative is compatible with the metric if

$$d\langle s_1, s_2\rangle = \langle\nabla_P s_1, s_2\rangle + \langle s_1, \nabla_P s_2\rangle$$

for any two sections s_1 and s_2 in $C^\infty(P/K_\infty, P\times_{K_\infty} z_0)$. There is a covariant derivative that is induced by a principal connection θ_P in $\Omega^1(P)\otimes\mathfrak{k}$ as follows. The derivative of the representation gives a map

$$d\rho\colon \mathfrak{k} \longrightarrow \mathfrak{so}(z_0)\subset \mathrm{End}(z_0),$$

which we also denote by ρ by abuse of notation. Note that for the representation (3.1.4) this is simply the map

$$\begin{aligned}\rho\colon \mathfrak{k} &\longrightarrow \mathfrak{so}(z_0)\\ X &\longmapsto X\big|_{z_0}\end{aligned}$$

since $\mathfrak{k} = \mathfrak{so}(z_0^\perp) \oplus \mathfrak{so}(z_0)$. Composing the principal connection with ρ defines an element

$$\rho(\theta_P) \in \Omega^1(P, \mathfrak{so}(z_0)).$$

In particular if $s\colon P/K_\infty \to P \times_{K_\infty} z_0$ is a section then we can identify it with a K_∞-invariant smooth map f_s in $C^\infty(P, z_0)^{K_\infty}$. Since $\rho(\theta_P)$ is an $\mathfrak{so}(z_0)$-valued form we can define

$$df_s + \rho(\theta_P) \cdot f_s \in \Omega^1(P, z_0).$$

Lemma 3.1.2. *The form $df_s + \rho(\theta_P) \cdot f_s$ is basic, hence gives a $P \times_{K_\infty} z_0$-valued form on P/K_∞. In particular, $\nabla_P = d + \rho(\theta_P)$ defines a covariant derivative on $P \times_{K_\infty} z_0$ which is compatible with the metric.*

Proof. See [BGV03, p. 24]. For the compatibility with the metric, it follows from the fact that the connection $\rho(\theta_P)$ is valued in $\mathfrak{so}(z_0) \subset \mathrm{End}(z_0)$ that

$$\langle \rho(\theta_P) f_{s_1}, f_{s_2} \rangle + \langle f_{s_1}, \rho(\theta_P) f_{s_2} \rangle = 0.$$

Hence

$$\langle \nabla_P s_1, s_2 \rangle + \langle s_1, \nabla_P s_2 \rangle = \langle df_{s_1}, f_{s_2} \rangle + \langle f_{s_1}, df_{s_2} \rangle = d\langle f_{s_1}, f_{s_2} \rangle = d\langle s_1, s_2 \rangle.$$

□

Curvature. A covariant derivative can be extended to a map

$$\nabla_P \colon \Omega^i(P/K_\infty, P \times_{K_\infty} z_0) \longrightarrow \Omega^{i+1}(P/K_\infty, P \times_{K_\infty} z_0) \tag{3.1.5}$$

by setting

$$\nabla_P(\omega \otimes s) := d\omega \otimes s + (-1)^i \omega \wedge \nabla_P(s),$$

where

$$\omega \otimes s \in \Omega^i(P/K_\infty) \otimes C^\infty(P \times_{K_\infty} z_0) \simeq \Omega^i(P/K_\infty, P \times_{K_\infty} z_0).$$

We define the *curvature* R_P in $\Omega^2(P, \mathfrak{k})$ by

$$R_P(X, Y) := [\theta_P(X), \theta_P(Y)] - \theta_P([X, Y]),$$

for two vector fields X and Y on P. It is basic by [BGV03, Proposition. 1.13] and composing with ρ gives an element

$$\rho(R_P) \in \Omega^2(P, \mathfrak{so}(z_0))_{bas}.$$

Then, for a section s in $C^\infty(P/K_\infty, P \times_{K_\infty} z_0)$ we have [BGV03, Proposition. 1.15]

$$\nabla_P^2 s = \rho(R_p)s \in \Omega^2(P/K_\infty, P \times_{K_\infty} z_0).$$

From now on we denote by ∇ and $\widetilde{\nabla}$ the covariant derivatives on E and $\widetilde{E}$ associated to θ and $\widetilde{\theta}$ defined in (3.1.2) and (3.1.3). Let R and $\widetilde{R}$ be their respective curvatures.

3.1.3 Pullback of bundles

The pullback of E by the projection map gives a canonical bundle

$$\pi^*E \coloneqq \left\{(e, e') \in E \times E \,|\, \pi(e) = \pi(e')\right\}$$

over E. We have the following diagram

$$\begin{CD} \pi^*E @>>> E \\ @VVV @VV{\pi}V \\ E @>{\pi}>> \mathbb{D}^+. \end{CD}$$

The projection also induces a pullback of the sections

$$\pi^* \colon \Omega^i(\mathbb{D}^+, E) \longrightarrow \Omega^i(E, \widetilde{E}),$$

and of the covariant derivative

$$\pi^*\nabla \colon C^\infty(E, \pi^*E) \longrightarrow \Omega^1(E, \pi^*E)$$

defined by the property

$$(\pi^*\nabla)(\pi^*s) = \pi^*(\nabla s).$$

By definition $([h_1, w_1], [h_2, w_2])$ is in π^*E if and only if $h_1^{-1}h_2$ is in K_∞. We have an $H(\mathbb{R})^+$-

equivariant morphism

$$\begin{aligned} \pi^* E &\longrightarrow \widetilde{E} \\ ([h_1, w_1], [h_2, w_2]) &\longrightarrow [(h_1, h_1^{-1} h_2 w_2), w_1]. \end{aligned} \tag{3.1.6}$$

This map is well defined, and has as inverse

$$\begin{aligned} \widetilde{E} &\longrightarrow \pi^* E \\ [(h, w_1), w_2] &\longrightarrow ([h, w_2], [h, w_1]). \end{aligned}$$

The second statement follows from $\widetilde{\theta} = \pi^*\theta$.

Proposition 3.1.3. *The map (3.1.6) is an isomorphism between $\widetilde{E}$ and π^*E, and this isomorphism identifies $\widetilde{\nabla}$ and $\pi^*\nabla$.*

3.1.4 A few operations on the vector bundles

We extend the K_∞-representation z_0 to $\bigwedge^j z_0$ by $k(w_1 \wedge \cdots \wedge w_j) = (kw_1) \wedge \cdots \wedge (kw_j)$. We consider the bundles $P \times_{K_\infty} \wedge^j z_0$ and $P \times_{K_\infty} \wedge z_0$ over P/K_∞, where $\bigwedge z_0 = \bigoplus_i \bigwedge^i z_0$. Denote the space of differential forms valued in $P \times_{K_\infty} \wedge^j z_0$ by

$$\Omega_P^{i,j} \coloneqq \Omega_P^i(P/K_\infty, P \times_{K_\infty} \wedge^j z_0) = \Omega_P^i(P/K_\infty) \otimes C^\infty(P/K_\infty, P \times_{K_\infty} \wedge^j z_0).$$

The total space of differential forms

$$\Omega(P/K_\infty, P \times_{K_\infty} \wedge z_0) = \bigoplus_{i,j} \Omega_P^{i,j}$$

is an (associative) bigraded $C^\infty(P/K_\infty)$-algebra where the product is defined by

$$\begin{aligned} \wedge \colon \Omega_P^{i,j} \times \Omega_P^{k,l} &\longrightarrow \Omega_P^{i+k,j+l} \\ (\omega \otimes s, \eta \otimes t) &\longmapsto (\omega \otimes s) \wedge (\eta \otimes t) \coloneqq (-1)^{jk} (\omega \wedge \eta) \otimes (s \wedge t). \end{aligned}$$

The exponential. This algebra structure allows us to define an *exponential map* by

$$\begin{aligned} \exp \colon \Omega(P/K_\infty, P \times_{K_\infty} \wedge z_0) &\longrightarrow \Omega(P/K_\infty, P \times_{K_\infty} \wedge z_0) \\ \omega &\longmapsto \exp(\omega) \coloneqq \sum_{k \geq 0} \frac{\omega^k}{k!}, \end{aligned}$$

where ω^k is the k-fold wedge product $\omega \wedge \cdots \wedge \omega$.

Remark 3.1.1. Suppose that ω and η commute. Then the binomial formula

$$(\omega+\eta)^k = \sum_{l=0}^{k} \binom{k}{l} \omega^l \eta^{k-l}$$

holds and one can show that $\exp(\omega+\eta) = \exp(\omega)+\exp(\eta)$ in the same way as for the real exponential map. In particular the diagonal subalgebra $\bigoplus \Omega_P^{i,i}$ is a commutative since for two forms ω and η in Ω_P we have

$$\omega \wedge \eta = (-1)^{\deg(\omega)+\deg(\eta)} \eta \wedge \omega$$

and similarly for two sections $s$and t in $\Omega^0(P/K, P \times_{K_\infty} z_0)$.

The Berezinian. The inner product $\langle -, - \rangle$ on z_0 can be extended to $\bigwedge z_0$ by

$$\langle \, \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k \, , \, \mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_l \, \rangle := \begin{cases} 0 & \text{if} \quad k \neq l, \\ \det \langle \mathbf{x}_i, \mathbf{y}_j \rangle_{i,j} & \text{if} \quad k = l. \end{cases}$$

The set

$$\{ \mathbf{e}_{p+i_1} \wedge \cdots \wedge \mathbf{e}_{p+i_k} \mid 1 \leq k \leq q, \; i_1 < i_2 < \cdots < i_k \}$$

is an orthonormal basis of $\bigwedge z_0$, since $\mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q}$ is an orthonormal basis of z_0. We define the *Berezin integral* $\int^B$ to be the orthogonal projection onto the top dimensional component

$$\begin{aligned} \int^B : \bigwedge z_0 &\longrightarrow \mathbb{R} \\ w &\longmapsto \langle w \, , \, \mathbf{e}_{p+1} \wedge \cdots \wedge \mathbf{e}_{p+q} \rangle. \end{aligned}$$

The Berezin integral can then be extended to

$$\begin{aligned} \int^B : \Omega(P/K_\infty, P \times_{K_\infty} \wedge z_0) &\longrightarrow \Omega(P/K_\infty) \\ \omega \otimes s &\longmapsto \omega \int^B s \end{aligned}$$

where $\int^B s$ in $C^\infty(P/K_\infty)$ is the composition of the section with the Berezinian in every fiber. Let $s_1, \ldots, s_q$ be a local orthonormal frame of $P \times_{K_\infty} z_0$, then

$$s_1 \wedge \cdots \wedge s_q \in C^\infty(P/K_\infty, \wedge^q P \times_{K_\infty} z_0)$$

is a global section and for α in $\Omega(P/K_\infty, P \times_{K_\infty} \wedge z_0)$ we have

$$\int^B \alpha = \langle \alpha, s_1 \wedge \cdots \wedge s_q \rangle.$$

The contraction. Finally, for every section s in $\Omega^{0,1}$ we can define the *contraction* $i(s)$ by

$$\begin{aligned} i(s)\colon \Omega_P^{i,j} &\longrightarrow \Omega_P^{i,j-1} \\ \omega \otimes s_1 \wedge \cdots \wedge s_j &\longmapsto \sum_{k=1}^{j} (-1)^{i+k-1} \langle s, s_k \rangle \omega \otimes s_1 \wedge \cdots \wedge \widehat{s_k} \wedge \cdots \wedge s_j \end{aligned}$$

and extended by linearity, where the symbol $\widehat{\cdot}$ means that we remove it from the product. Note that when $j = 0$ then $i(s)$ is defined to be 0. It defines a derivation on $\oplus \widetilde{\Omega}^{i,j}$ that satisfies

$$i(s)(\alpha \wedge \alpha') = (i(s)\alpha) \wedge \alpha' + (-1)^{i+j} \alpha \wedge (i(s)\alpha')$$

for α in $\widetilde{\Omega}^{i,j}$ and α' in $\widetilde{\Omega}^{k,l}$.

3.1.5 The Mathai-Quillen construction

Let us now recall the construction of the Mathai-Quillen form U_{MQ} in $\Omega^q(E)$. As earlier let $\widetilde{E} = (H(\mathbb{R})^+ \times z_0) \times_{K_\infty} z_0$ and $\wedge^j \tilde{E} := (H(\mathbb{R})^+ \times z_0) \times_{K_\infty} \wedge^j z_0$. We set

$$\begin{aligned} \Omega^{i,j} &:= \Omega^i(\mathbb{D}^+, \wedge^j E) \\ \widetilde{\Omega}^{i,j} &:= \Omega^i(E, \wedge^j \widetilde{E}). \end{aligned}$$

First consider the tautological section $\mathbf{s}$ in $\widetilde{\Omega}^{0,1}$ defined by

$$\begin{aligned} \mathbf{s}\colon E &\longrightarrow \widetilde{E} \\ [h, w] &\longmapsto \mathbf{s}[h, w] := [(g, w), w]. \end{aligned}$$

Composing with the norm induced from the inner product we get an element

$$\|\mathbf{s}\|^2 \in \widetilde{\Omega}^{0,0}.$$

The representation ρ on z_0 induces a representation on $\wedge^i z_0$ that we also denote by ρ. The derivative at the identity gives a map

$$\rho\colon \mathfrak{k} \longrightarrow \mathfrak{so}(\wedge^i z_0).$$

The connection form $\rho(\widetilde{\theta})$ in $\Omega^1(H(\mathbb{R})^+ \times z_0, \wedge^j z_0)$ defines a covariant derivative

$$\widetilde{\nabla}\colon \widetilde{\Omega}^{0,j} \longrightarrow \widetilde{\Omega}^{1,j}.$$

We can extend it to

$$\widetilde{\nabla}\colon \widetilde{\Omega}^{i,j} \longrightarrow \widetilde{\Omega}^{i+1,j},$$

by setting

$$\widetilde{\nabla}(\omega \otimes s) \coloneqq d\omega \otimes s + (-1)^i \omega \wedge \widetilde{\nabla}(s),$$

as in (3.1.5). The connection on $\widetilde{\Omega}^{i,j}$ is compatible with the metric.

Finally, the covariant derivative $\widetilde{\nabla}$ defines a derivation on $\oplus\widetilde{\Omega}^{i,j}$ that satisfies

$$\widetilde{\nabla}(\alpha \wedge \alpha') = (\widetilde{\nabla}\alpha) \wedge \alpha' + (-1)^{i+j} \alpha \wedge (\widetilde{\nabla}\alpha')$$

for any α in $\widetilde{\Omega}^{i,j}$ and α' in $\widetilde{\Omega}^{k,l}$.

Taking the derivative of the tautological section gives an element

$$\widetilde{\nabla}\mathbf{s} = d\mathbf{s} + \rho(\widetilde{\theta})\mathbf{s} \in \widetilde{\Omega}^{1,1}.$$

Let $\mathfrak{so}(\widetilde{E})$ denote the bundle $(H(\mathbb{R})^+ \times z_0) \times_K \mathfrak{so}(z_0)$ and consider the curvature $\rho(\widetilde{R})$ in $\Omega^2(\widetilde{E}, \mathfrak{so}(\widetilde{E}))$. We have an isomorphism

$$\begin{aligned} T^{-1}\big|_{z_0} \colon \mathfrak{so}(z_0) &\longrightarrow \wedge^2 z_0 \\ A &\longmapsto \sum_{i<j} \langle A e_i, e_j \rangle e_i \wedge e_j. \end{aligned}$$

The inverse sends $v \wedge w$ to the endomorphism $u \mapsto \langle v, u \rangle w - \langle w, u \rangle v$, and is the isomorphism from (2.2.2) restricted to z_0. Note that we have

$$T(v \wedge w)u = \iota(u) v \wedge w. \tag{3.1.7}$$

Using this isomorphism we can also identify $\mathfrak{so}(\widetilde{E})$ and $\wedge^2 \widetilde{E}$ so that we can view the curvature as an element

$$\rho(\widetilde{R}) \in \widetilde{\Omega}^{2,2}.$$

Lemma 3.1.4. *The form* $\omega \coloneqq 2\pi \|\mathbf{s}\|^2 + 2\sqrt{\pi}\widetilde{\nabla}\mathbf{s} - \rho(\widetilde{R})$ *lying in* $\widetilde{\Omega}^{0,0} \oplus \widetilde{\Omega}^{1,1} \oplus \widetilde{\Omega}^{2,2}$ *is annihilated* *by* $\widetilde{\nabla} + 2\sqrt{\pi}i(\mathbf{s})$. *Moreover*

$$d\int^B \alpha = \int^B \widetilde{\nabla}\alpha,$$

for every form α *in* $\widetilde{\Omega}^{i,j}$. *Hence* $\int^B exp(-\omega)$ *is a closed form.*

Proof. We have

$$\begin{aligned}&\left(\widetilde{\nabla} + 2\sqrt{\pi}i(\mathbf{s})\right)\left(2\pi\|\mathbf{s}\|^2 + 2\sqrt{\pi}\widetilde{\nabla}\mathbf{s} - \rho(\widetilde{R})\right)\\ &= 2\pi\widetilde{\nabla}\|\mathbf{s}\|^2 + 4\pi^{\frac{3}{2}}i(\mathbf{s})\|\mathbf{s}\|^2 + 2\sqrt{\pi}\widetilde{\nabla}^2\mathbf{s} + 4\pi i(\mathbf{s})\widetilde{\nabla}\mathbf{s} - \widetilde{\nabla}\rho(\widetilde{R}) - 2\sqrt{\pi}i(\mathbf{s})\rho(\widetilde{R}).\end{aligned}$$

It vanishes because we have the following:

- $i(\mathbf{s})\|\mathbf{s}\|^2 = 0$ since $\|\mathbf{s}\| \in \widetilde{\Omega}^{0,0}$,
- $\widetilde{\nabla}\rho(\widetilde{R}) = 0$ by Bianchi's identity,
- $\widetilde{\nabla}\|\mathbf{s}\|^2 = 2\langle\widetilde{\nabla}\mathbf{s}, \mathbf{s}\rangle = -2i(\mathbf{s})\widetilde{\nabla}\mathbf{s}$,
- $\widetilde{\nabla}^2\mathbf{s} = \langle\rho(\widetilde{R}), \mathbf{s}\rangle = i(\mathbf{s})\rho(\widetilde{R})$.

For the last point we used (3.1.7) where we view $\rho(\widetilde{R})$ as an element of $\Omega^2(E, \mathfrak{so}(\widetilde{E}))$, respectively of $\Omega^2(E, \wedge^2\widetilde{E})$.

Let $s_1 \wedge \cdots \wedge s_q$ in $C^\infty(E, \wedge^q\widetilde{E})$ be a global section where $s_1, \ldots, s_q$ is a local orthonormal frame for $\widetilde{E}$. Then for α in $\widetilde{\Omega}^{i,j}$ we have

$$\int^B \alpha = \langle\alpha, s_1 \wedge \cdots \wedge s_q\rangle.$$

This vanishes if $j < q$ so we can assume $\alpha \in \widetilde{\Omega}^{i,q}$. If we write $\alpha = \beta s_1 \wedge \cdots \wedge s_q$ for some β in $\Omega^i(E)$ then

$$\int^B \alpha = \beta.$$

On the other hand, since the connection on $\widetilde{\Omega}^{i,q}$ is compatible with the metric, we have

$$0 = d\langle s_1 \wedge \cdots \wedge s_q, s_1 \wedge \cdots \wedge s_q\rangle = 2\langle\widetilde{\nabla}(s_1 \wedge \cdots \wedge s_q), s_1 \wedge \cdots \wedge s_q\rangle.$$

Then we have

$$\begin{aligned}\int^B \widetilde{\nabla}\alpha &= \langle\widetilde{\nabla}\alpha, s_1 \wedge \cdots \wedge s_q\rangle\\ &= \langle d\beta \otimes s_1 \wedge \cdots \wedge s_q + (-1)^i\beta \wedge \widetilde{\nabla}(s_1 \wedge \cdots \wedge s_q), s_1 \wedge \cdots \wedge s_q\rangle\\ &= d\beta = d\int^B \alpha.\end{aligned}$$

Finally, since $\widetilde{\nabla} + 2\sqrt{\pi} i(\mathbf{s})$ is a derivation that annihilates ω we have

$$\left(\widetilde{\nabla} + 2\sqrt{\pi} i(\mathbf{s})\right) \omega^k = 0$$

for $k > 0$ and

$$\begin{aligned} d \int^B \exp(-\omega) &= \int^B \widetilde{\nabla} \exp(-\omega) \\ &= \int^B \left(\widetilde{\nabla} + 2\sqrt{\pi} i(\mathbf{s})\right) \exp(-\omega) = 0. \end{aligned}$$

□

Proposition 3.1.5. *The form*

$$U_{MQ} := (-1)^{\frac{q(q+1)}{2}} (2\pi)^{-\frac{q}{2}} \int^B \exp\left(-2\pi \|\mathbf{s}\|^2 - 2\sqrt{\pi}\widetilde{\nabla}\mathbf{s} + \rho(\widetilde{R})\right) \in \Omega^q(E)$$

is a Thom form.

Proof. From the previous lemma it follows that the form is closed, it remains to show that its integral along the fibers is 1. The restriction of the form U_{MQ} along the fiber $\pi^{-1}(eK_\infty)$ is given by

$$\begin{aligned} U_{MQ} &= (-1)^{\frac{q(q+1)}{2}} (2\pi)^{-\frac{q}{2}} e^{-2\pi\|\mathbf{s}\|^2} \int^B \exp(-2\sqrt{\pi} d\mathbf{s}) \\ &= (-1)^{\frac{q(q+1)}{2}} 2^{\frac{q}{2}} e^{-2\pi\|\mathbf{s}\|^2} (-1)^q \int^B (dx_1 \otimes \mathbf{e}_1) \wedge \cdots \wedge (dx_q \otimes \mathbf{e}_q) \\ &= 2^{\frac{q}{2}} e^{-2\pi\|\mathbf{s}\|^2} dx_1 \wedge \cdots \wedge dx_q, \end{aligned}$$

and its integral over the fiber $\pi^{-1}(eK_\infty)$ is equal to 1. □

3.1.6 Transgression form

For $t > 0$ consider the map $t\colon E \longrightarrow E$ given by multiplication by t in the fibers. Consider the K_∞-invariant vector field

$$X := \sum_{i=1}^q x_i \frac{\partial}{\partial x_i}$$

on $H(\mathbb{R})^+ \times \mathbb{R}^q$. Since it is K_∞-invariant, it induces a vector field on E. We define the *transgression form* $\psi := \iota_X U_{MQ}$ in $\Omega^{q-1}(E)$, where ι_X is the interior product.

Proposition 3.1.6 (Transgression formula)**.** *The transgression satisfies*

$$\left(\frac{d}{dt}t^*U_{MQ}\right)_{t=t_0} = -\frac{1}{t_0}d(t_0^*\psi).$$

Proof. Let us view the multiplication map by t as a map

$$\begin{aligned} m\colon E\times\mathbb{R}_{>0} &\longrightarrow E \\ (e,t) &\longmapsto et. \end{aligned}$$

The differential $\tilde{d}$ on $E\times\mathbb{R}_{>0}$ splits as $\tilde{d} = d + d_{\mathbb{R}_{>0}}$. Since U_{MQ} is closed (hence its pullback) we have

$$0 = \tilde{d}(m^*U_{MQ}) = d(m^*U_{MQ}) + \frac{d}{dt}(m^*U_{MQ})dt. \tag{3.1.8}$$

Moreover the pushforward of the vector field $t\frac{\partial}{\partial t}$ by m is X, hence for the contraction we have

$$\iota_{\frac{\partial}{\partial t}}m^*U_{MQ} = \frac{1}{t}m^*\iota_X U_{MQ}.$$

Since the differential d is independent of t it commutes with the contraction $\iota_{\frac{\partial}{\partial t}}$. Combining with (3.1.8) yields

$$\frac{d}{dt}(m^*U_{MQ}) = -\frac{1}{t}d(m^*\psi).$$

Finally, pulling back by the section $t_0\colon E\longmapsto E\times\mathbb{R}_{>0}$ sending e to (e,t_0) gives the desired formula. □

Let $\Gamma_{\mathbf{x}}$ be the stabilizer of $\mathbf{x}$ in Γ, which acts on the left on E. By the $H(\mathbb{R})^+$-invariance (hence $\Gamma_{\mathbf{x}}$-invariance) of U_{MQ}, it is also a form in $\Omega^q(\Gamma_{\mathbf{x}}\backslash E)$. Let S_0 denote the image $\Gamma_{\mathbf{x}}\backslash E_0$ of the zero section in $\Gamma_{\mathbf{x}}\backslash E$.

Proposition 3.1.7. *The form U_{MQ} represents the Poincaré dual of S_0 in $\Gamma_{\mathbf{x}}\backslash E$.*

Sketch of proof. For $0<t_1<t_2$ we have

$$\begin{aligned} t_2^*U_{MQ} - t_1^*U_{MQ} &= \int_{t_1}^{t_2}\left(\frac{d}{dt}t^*U_{MQ}\right)dt \\ &= -\int_{t_1}^{t_2} d(t^*\psi)\frac{dt}{t} \\ &= -d\int_{t_1}^{t_2} t^*\psi\frac{dt}{t} \end{aligned}$$

so that $t_2^* U_{MQ}$ and $t_1^* U_{MQ}$ represent the same cohomology class in $H^q(\Gamma_{\mathbf{x}} \backslash E)$. Then, one can show that

$$\lim_{t \to \infty} t^* U_{MQ} = \delta_{S_0}$$

where δ_{S_0} is the current of integration along S_0. Hence if ω is a compactly supported form in $\Omega_c^m(E)$, where m is the dimension of $\mathbb{D}^+$, then

$$\begin{aligned}\int_{\Gamma_{\mathbf{x}} \backslash E} U_{MQ} \wedge \omega &= \lim_{t \to \infty} \int_{\Gamma_{\mathbf{x}} \backslash E} t^* U_{MQ} \wedge \omega \\ &= \int_{S_0} \omega.\end{aligned}$$

□

3.2 Computation of the Mathai-Quillen form

Let us now compute explicitely the Mathai-Quillen form constructed in 3.1.5.

3.2.1 The section $s_{\mathbf{x}}$

Suppose that $\mathbf{x}$ is a vector in $X_{\mathbb{R}}$ with $Q(\mathbf{x}, \mathbf{x}) > 0$. Then

$$\mathbb{D}_{\mathbf{x}}^+ := \left\{ z \in \mathbb{D}^+ \mid z \subset \mathbf{x}^{\perp} \right\}.$$

Let $\mathrm{pr}\colon X_{\mathbb{R}} \longrightarrow z_0$ be the orthogonal projection on the plane z_0. Consider the section

$$\begin{aligned} s_{\mathbf{x}}\colon \mathbb{D}^+ &\longrightarrow E \\ z &\longmapsto [h_z, \mathrm{pr}(h_z^{-1}\mathbf{x})], \end{aligned}$$

where $h_z \in H(\mathbb{R})^+$ is any element sending z_0 to z. Let us denote by l_h the left action of $h \in H(\mathbb{R})^+$ on $\mathbb{D}^+$ and E.

Proposition 3.2.1. *The section $s_{\mathbf{x}}$ is well-defined, $\Gamma_{\mathbf{x}}$-equivariant and its zero locus is precisely $\mathbb{D}_{\mathbf{x}}^+$.*

Proof. The section is well-defined, since replacing h by hk gives

$$s_{\mathbf{x}}(z) = [h_z k, \mathrm{pr} k^{-1} h_z^{-1} \mathbf{x}] = [h_z k, k^{-1} \mathrm{pr} h_z^{-1} \mathbf{x}] = [h, \mathrm{pr}(h_z^{-1}\mathbf{x})] = s_{\mathbf{x}}(z).$$

Suppose that $z = hz_0$ is in the zero locus of $s_{\mathbf{x}}$, that is to say $\mathrm{pr}(h^{-1}\mathbf{x}) = 0$. Then $h^{-1}\mathbf{x}$ is in $z_0^{\perp}$, which is equivalent to hz_0 being a subspace of $\mathbf{x}^\perp$. Hence the zero locus of $s_\mathbf{x}$ is exactly $\mathbb{D}_\mathbf{x}^+$. For the equivariance, note that we have

$$s_\mathbf{x} \circ l_h(z) = [hh_z, \mathrm{pr}(h_z^{-1}h^{-1}\mathbf{x})] = l_h \circ s_{h^{-1}\mathbf{x}}(z).$$

Hence if $\gamma \in \Gamma_\mathbf{x}$ we have $s_\mathbf{x} \circ l_\gamma = l_\gamma \circ s_\mathbf{x}$. □

We define the pullback of the Mathai-Quillen form by $s_\mathbf{x}$

$$\varphi^0(\mathbf{x}) := s_\mathbf{x}^* U_{MQ} \in C^\infty(\mathbb{R}^{p+q}) \otimes \Omega^q(\mathbb{D})^+.$$

It is only rapidly decrasing on $\mathbb{R}^q$, and in order to make it rapidly decreasing everywhere we set

$$\varphi(\mathbf{x}) := e^{-\pi Q(\mathbf{x},\mathbf{x})}\varphi^0(\mathbf{x}) \in \mathscr{S}(\mathbb{R}^{p+q}) \otimes \Omega^q(\mathbb{D})^+.$$

Proposition 3.2.2. *1. The form $\varphi^0(\mathbf{x})$ is equal to*

$$(-1)^{\frac{q(q+1)}{2}}(2\pi)^{-\frac{q}{2}} \exp\left(2\pi Q\big|_{z_0}(\mathbf{x},\mathbf{x})\right)\int^B \exp\left(-2\sqrt{\pi}\nabla s_\mathbf{x} + \rho(R)\right) \in \Omega^q(\mathbb{D}^+).$$

2. *It satisfies $l_h^*\varphi^0(\mathbf{x}) = \varphi^0(h^{-1}\mathbf{x})$, hence*

$$\varphi^0 \in \left[\Omega^q(\mathbb{D}^+) \otimes C^\infty(\mathbb{R}^{p+q})\right]^{H(\mathbb{R})^+}.$$

3. *In particular we can view it as form on $\Gamma_\mathbf{x}\backslash\mathbb{D}^+$, and as such it represents a Poincaré dual to $\Gamma_\mathbf{x}\backslash\mathbb{D}_\mathbf{x}^+$.*

Proof. 1. Recall that $\widetilde{\nabla} = \pi^*\nabla$ and $\widetilde{R} = \pi^*R$. We pullback by $s_\mathbf{x}$

$$\begin{array}{ccc} E \simeq s_\mathbf{x}^*\widetilde{E} & \longrightarrow & \widetilde{E} \\ \downarrow & & \downarrow \pi \\ \mathbb{D}^+ & \xrightarrow{s_\mathbf{x}} & E. \end{array}$$

The pullback connection $s_\mathbf{x}^*\widetilde{\nabla} = \nabla$ satisfies

$$s_\mathbf{x}^*(\widetilde{\nabla}\mathbf{s}) = (s_\mathbf{x}^*\widetilde{\nabla})(s_\mathbf{x}^*\mathbf{s}) = \nabla s_\mathbf{x},$$

since $s_\mathbf{x}^*\mathbf{s} = s_\mathbf{x}$. We also have $s_\mathbf{x}^*\widetilde{R} = R$ and

$$s_\mathbf{x}^*\|\mathbf{s}\|^2 = \|s_\mathbf{x}\|^2 = \langle s_\mathbf{x}, s_\mathbf{x}\rangle = -Q\big|_{z_0}(\mathbf{x},\mathbf{x}).$$

The expression for φ^0 then follows from the fact that exp and $s_{\mathbf{x}}^*$ commute.

2. The bundle E is $H(\mathbb{R})^+$ equivariant, where the left action on E is $l_h[h_z, \mathbf{x}] = [hh_z, \mathbf{x}]$. By construction the Mathai-Quillen is $H(\mathbb{R})^+$-invariant, so $l_h^* U_{MQ} = U_{MQ}$. On the other hand we also have
$$s_{\mathbf{x}} \circ l_h(z) = l_h \circ s_{h^{-1}\mathbf{x}}(z),$$
and thus $l_h^* \varphi^0(\mathbf{x}) = l_h^* s_{\mathbf{x}}^* U_{MQ} = \varphi^0(h^{-1}\mathbf{x})$.

3. Since $s_{\mathbf{x}}$ is $\Gamma_{\mathbf{x}}$-equivariant we view it as a section
$$s_{\mathbf{x}} \colon \Gamma_{\mathbf{x}} \backslash \mathbb{D}^+ \longrightarrow \Gamma_{\mathbf{x}} \backslash E,$$
whose zero locus is precisely $\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+_{\mathbf{x}}$. By Proposition 3.1.7 the Thom form U_{MQ} is a Poincaré dual of the zero section $S_0 \subset \Gamma_{\mathbf{x}} \backslash E$. Let $S_{\mathbf{x}} \coloneqq s_{\mathbf{x}}(\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+) \subset \Gamma_{\mathbf{x}} \backslash E$ be the image of $S_{\mathbf{x}}$. For $\omega \in \Omega_c^{pq-q}(\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+)$ we have
$$\begin{aligned} \int_{\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+} \varphi^0(\mathbf{x}) \wedge \omega &= \int_{\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+} s_{\mathbf{x}}^* \left(U_{MQ} \wedge \pi^* \omega \right) \\ &= \int_{S_{\mathbf{x}}} U_{MQ} \wedge \pi^* \omega \\ &= \int_{S_{\mathbf{x}} \cap S_0} \pi^* \omega \\ &= \int_{\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+_{\mathbf{x}}} \omega. \end{aligned}$$
The last step follows from the fact $\pi^{-1}(s_{\mathbf{x}} \cap S_0) = \Gamma_{\mathbf{x}} \backslash \mathbb{D}^+_{\mathbf{x}}$.

□

3.2.2 Computation at the identity

From now on we identify $X_{\mathbb{R}}$ with $\mathbb{R}^{p+q}$ by the orthonormal basis of (3.1.1), and let z_0 be the negative q-plane spanned by the vectors $\mathbf{e}_{p+1}, \cdots, \mathbf{e}_{p+q}$. Hence we identify z_0 with $\mathbb{R}^q$ and the quadratic form is
$$Q\big|_{z_0}(\mathbf{x}, \mathbf{x}) = -\sum_{\mu=p+1}^{p+q} x_\mu^2,$$

where $x_{p+1}, \dots, x_{p+q}$ are the coordinates of $\mathbf{x}$. As in (2.2.4) we have an isomorphism

$$\begin{aligned} \left[\Omega^q(\mathbb{D}^+) \otimes C^\infty(\mathbb{R}^{p+q})\right]^{H(\mathbb{R})^+} &\longrightarrow \left[\bigwedge^q \mathfrak{p}^* \otimes C^\infty(\mathbb{R}^{p+q})\right]^{K_\infty} \\ \varphi &\longmapsto \varphi\big|_e \end{aligned}$$

by evaluating at the basepoint eK_∞ corresponding to z_0 in $\mathbb{D}^+$. We will now compute $\varphi^0\big|_e$.

Let $f_{\mathbf{x}}(h)$ in $C^\infty(H(\mathbb{R})^+, z_0)^{K_\infty}$ be the map associated to the section $s_{\mathbf{x}}$, as in Proposition 3.1.1. It is defined by

$$f_{\mathbf{x}}(h) = \mathrm{pr}(h^{-1}\mathbf{x}).$$

Then $df_{\mathbf{x}} + \rho(\theta)f_{\mathbf{x}}$ is the horizontal lift of $\nabla s_{\mathbf{x}}$. Let X be a vector in $\mathfrak{g}$ and let $X_{\mathfrak{p}}$ and $X_{\mathfrak{k}}$ be its components with respect to the splitting of $\mathfrak{g}$ as $\mathfrak{p} \oplus \mathfrak{k}$. We have

$$(df_{\mathbf{x}} + \rho(\theta)f_{\mathbf{x}})_e(X) = d_e f_{\mathbf{x}}(X_{\mathfrak{p}}).$$

In particular we can evaluate it on the basis $X_{\alpha\mu}$ and get:

$$\begin{aligned} d_e f_{\mathbf{x}}(X_{\alpha\mu}) &= \frac{d}{dt}\bigg|_{t=0} f_{\mathbf{x}}(\exp tX_{\alpha\mu}) \\ &= -\,\mathrm{pr}(X_{\alpha\mu}\mathbf{x}) \\ &= -\,\mathrm{pr}(x_\mu \mathbf{e}_\alpha + x_\alpha \mathbf{e}_\mu) \\ &= -x_\alpha \mathbf{e}_\mu. \end{aligned}$$

So as an element of $\mathfrak{p}^* \otimes z_0$ we can write

$$d_e f_{\mathbf{x}} = -\sum_{\mu=p+1}^{p+q}\left(\sum_{\alpha=1}^{p} x_\alpha \omega_{\alpha\mu}\right) \otimes \mathbf{e}_\mu = -\sum_{\alpha=1}^{p} x_\alpha \eta_\alpha,$$

with

$$\eta_\alpha \coloneqq \sum_{\mu=p+1}^{p+q} \omega_{\alpha\mu} \otimes \mathbf{e}_\mu \in \Omega^{1,1}.$$

Proposition 3.2.3. *Let $\rho(R)_e$ in $\wedge^2\mathfrak{p}^* \otimes \mathfrak{so}(z_0)$ be the curvature at the identity. Then after identifying $\mathfrak{so}(z_0)$ with $\wedge^2 z_0$ we have*

$$\rho(R)_e = -\frac{1}{2}\sum_{\alpha=1}^{p} \eta_\alpha^2 \in \wedge^2\mathfrak{p}^* \otimes \wedge^2 z_0,$$

where $\eta_\alpha^2 = \eta_\alpha \wedge \eta_\alpha$.

Proof. Using the relation $E_{ij}E_{kl} = \delta_{il}E_{kj}$ one can show that

$$[X_{\alpha\mu}, X_{\beta\nu}] = \delta_{\mu\nu}X_{\alpha\beta} + \delta_{\alpha\beta}X_{\mu\nu}$$

for two vectors $X_{\alpha\nu}$ and $X_{\beta\mu}$ in $\mathfrak{p}$. Hence we have

$$\begin{aligned} R_e(X_{\alpha\nu} \wedge X_{\beta\mu}) &= [\theta(X_{\alpha\nu}), \theta(X_{\beta\mu})] - \theta([X_{\alpha\nu}, X_{\beta\mu}]) \\ &= -\theta([X_{\alpha\nu}, X_{\beta\mu}]) \\ &= -p\left(\delta_{\alpha\beta}X_{\nu\mu} + \delta_{\nu\mu}X_{\alpha\beta}\right) \\ &= -\delta_{\alpha\beta}X_{\nu\mu}. \end{aligned}$$

On the other hand, since $\eta_i(X_{jr}) = \delta_{ij}\mathbf{e}_r$, we also have

$$\begin{aligned} \sum_{i=1}^{p} \eta_i^2(X_{\alpha\nu} \wedge X_{\beta\mu}) &= \sum_{i=1}^{p} \eta_i(X_{\alpha\nu}) \wedge \eta_i(X_{\beta\mu}) - \eta_i(X_{\beta\mu}) \wedge \eta_i(X_{\alpha\nu}) \\ &= 2\delta_{\alpha\beta}\mathbf{e}_\nu \wedge \mathbf{e}_\mu. \end{aligned}$$

The lemma follows since $\rho(X_{\nu\mu}) = T(e_\nu \wedge e_\mu)$ in $\mathfrak{so}(z_0)$, because

$$Q(\rho(X_{\nu\mu})\mathbf{e}_\nu, \mathbf{e}_\mu)\mathbf{e}_\nu \wedge \mathbf{e}_\mu = -Q(\mathbf{e}_\mu, \mathbf{e}_\mu)\mathbf{e}_\nu \wedge \mathbf{e}_\mu = \mathbf{e}_\nu \wedge \mathbf{e}_\mu.$$

□

Using the fact that the exponential satisfies $\exp(\omega+\eta) = \exp(\omega)\exp(\eta)$ on the subalgebra $\bigoplus \Omega^{i,i}$ - see Remark 3.1.1 - we can write $\varphi^0\big|_e(\mathbf{x})$ as

$$(-1)^{\frac{q(q+1)}{2}}(2\pi)^{-\frac{q}{2}} \exp\left(2\pi Q\big|_{z_0}(\mathbf{x},\mathbf{x})\right) \int^B \prod_{\alpha=1}^{p} \exp\left(2\sqrt{\pi}x_\alpha\eta_\alpha - \frac{1}{2}\eta_\alpha^2\right). \tag{3.2.1}$$

We define the *n-th Hermite polynomial* by

$$H_n(x) := \left(2x - \frac{d}{dx}\right) \cdot 1 \in \mathbb{R}[x].$$

The first few examples are $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$...

Lemma 3.2.4. *Let* $\eta \in \bigoplus \Omega^{i,i}$. *Then*

$$\exp(2x\eta - \eta^2) = \sum_{n\geq 0} \frac{1}{n!} H_n(x)\eta^n,$$

where H_k *is the* k*-th Hermite polynomial.*

Proof. Since η and η^2 are in $\bigoplus \Omega^{i,i}$, they commute and we can use the binomial formula:

$$\begin{aligned}
\exp(2x\eta - \eta^2) &= \sum_{k\geq 0} \frac{1}{k!} \left(2x\eta - \eta^2\right)^k \\
&= \sum_{k\geq 0} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (2x\eta)^{k-l} \left(-\eta^2\right)^l \\
&= \sum_{k\geq 0} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (2x)^{k-l} (-1)^l \eta^{l+k} \\
&= \sum_{n\geq 0} P_n(x)\eta^n,
\end{aligned}$$

where

$$P_n(x) := \sum_{\substack{0\leq l\leq k\leq n \\ k+l=n}} \frac{(-1)^l}{l!(k-l)!} (2x)^{k-l}.$$

The conditions on k and l imply that $n \leq 2k$. First suppose that n is even, then we have $\frac{n}{2} \leq k \leq n$ and the sum above can be written

$$\begin{aligned}
\sum_{k=\frac{n}{2}}^{n} \frac{(-1)^{n-k}}{(n-k)!(2k-n)!} (2x)^{2k-n} &= \sum_{m=0}^{\frac{n}{2}} \frac{(-1)^{\frac{n}{2}-m}}{(\frac{n}{2}-m)!(2m)!} (2x)^{2m} \\
&= \frac{1}{n!} H_n(x),
\end{aligned}$$

where in the second step we set $m = k - \frac{n}{2}$. If n is odd then $\frac{n+1}{2} \leq k \leq n$ and the sum can be written

$$\begin{aligned}
\sum_{k=\frac{n+1}{2}}^{n} \frac{(-1)^{n-k}}{(n-k)!(2k-n)!} (2x)^{2k-n} &= \sum_{m=0}^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2}-m}}{(\frac{n-1}{2}-m)!(2m+1)!} (2x)^{2m+1} \\
&= \frac{1}{n!} H_n(x).
\end{aligned}$$

□

Applying Lemma 3.2.4 to (3.2.1) we get

$$\int^B \prod_{\alpha=1}^p \exp\left(2\sqrt{\pi}x_\alpha\eta_\alpha - \frac{1}{2}\eta_\alpha^2\right) = \int^B \prod_{\alpha=1}^p \exp\left(2\sqrt{2\pi}x_\alpha\frac{\eta_\alpha}{\sqrt{2}} - \left(\frac{\eta_\alpha}{\sqrt{2}}\right)^2\right)$$
$$= \int^B \prod_{\alpha=1}^p \sum_{n\geq 0} \frac{2^{-n/2}}{n!} H_n\left(\sqrt{2\pi}x_\alpha\right)\eta_\alpha^n.$$

Expanding the product gives

$$\sum_{n_1,\dots,n_p} \frac{2^{-\frac{n_1+\cdots+n_p}{2}}}{n_1!\cdots n_p!} H_{n_1}\left(\sqrt{2\pi}x_1\right)\cdots H_{n_p}\left(\sqrt{2\pi}x_p\right)\int^B \eta_1^{n_1}\wedge\cdots\wedge\eta_p^{n_p}. \tag{3.2.2}$$

If $n_1+\cdots+n_p \neq q$, then the Berezinian of $\eta_1^{n_1}\wedge\cdots\wedge\eta_p^{n_p}$ is 0 and (3.2.2) is equal to

$$2^{-\frac{q}{2}} \sum_{n_1+\cdots+n_p=q} \frac{H_{n_1}\left(\sqrt{2\pi}x_1\right)\cdots H_{n_p}\left(\sqrt{2\pi}x_p\right)}{n_1!\cdots n_p!}\int^B \eta_1^{n_1}\wedge\cdots\wedge\eta_p^{n_p}.$$

Note that

$$\begin{aligned}\eta_\alpha^{n_\alpha} &= \left(\sum_{\mu=p+1}^{p+q}\omega_{\alpha\mu}\otimes\mathbf{e}_\mu\right)^{n_\alpha}\\ &= \sum_{\mu_1,\dots,\mu_{n_\alpha}}(\omega_{\alpha\mu_1}\otimes\mathbf{e}_{\mu_1})\wedge\cdots\wedge(\omega_{\alpha\mu_{n_\alpha}}\otimes\mathbf{e}_{\mu_{n_\alpha}})\\ &= n_\alpha!\sum_{\mu_1<\cdots<\mu_{n_\alpha}}(\omega_{\alpha\mu_1}\otimes\mathbf{e}_{\mu_1})\wedge\cdots\wedge(\omega_{\alpha\mu_{n_\alpha}}\otimes\mathbf{e}_{\mu_{n_\alpha}}),\end{aligned}$$

where the sums are over all $p+1\leq\mu_i\leq p+q$. If $n_1+\cdots+n_p=q$ we have

$$\begin{aligned}\int^B \eta_1^{n_1}\wedge\cdots\wedge\eta_p^{n_p} &= \int^B\prod_{\alpha=1}^p\left(\sum_{\mu=p+1}^{p+q}\omega_{\alpha\mu}\otimes\mathbf{e}_\mu\right)^{n_\alpha}\\ &= \int^B\prod_{\alpha=1}^p n_\alpha!\sum_{\mu_1<\cdots<\mu_{n_\alpha}}(\omega_{\alpha\mu_1}\otimes\mathbf{e}_{\mu_1})\wedge\cdots\wedge(\omega_{\alpha\mu_{n_\alpha}}\otimes\mathbf{e}_{\mu_{n_\alpha}})\\ &= n_1!\cdots n_p!\sum\int^B(\omega_{\alpha(p+1)}\otimes\mathbf{e}_1)\wedge\cdots\wedge(\omega_{\alpha(p+q)}\otimes\mathbf{e}_q)\\ &= (-1)^{\frac{q(q+1)}{2}}n_1!\cdots n_p!\sum\omega_{\alpha_1(p+1)}\wedge\cdots\wedge\omega_{\alpha_q(p+q)},\end{aligned}$$

where the sums in the last two lines go over all tuples $\underline{\alpha}=(\alpha_1,\dots,\alpha_q)$ such that $1\leq\alpha_i\leq p$ and the value $1 \leq i \leq p$ appears exactly n_i-times in $\underline{\alpha}$. Hence $\varphi^0\big|_e(\mathbf{x})$ is equal to

$$2^{-q}\pi^{-\frac{q}{2}} \sum \omega_{\alpha_1 p+1} \wedge \cdots \wedge \omega_{\alpha_q p+q} \otimes H_{n_1}\left(\sqrt{2\pi}x_1\right) \cdots H_{n_p}\left(\sqrt{2\pi}x_p\right) \exp\left(2\pi Q\big|_{z_0}(\mathbf{x},\mathbf{x})\right) \tag{3.2.3}$$

After multiplying by $\exp\left(-\pi Q(\mathbf{x},\mathbf{x})\right)$ we get that $\varphi\big|_e(\mathbf{x})$ is given by

$$2^{-q}\pi^{-\frac{q}{2}} \sum \omega_{\alpha_1 p+1} \wedge \cdots \wedge \omega_{\alpha_q p+q} \otimes H_{n_1}\left(\sqrt{2\pi}x_1\right) \cdots H_{n_p}\left(\sqrt{2\pi}x_p\right) \exp\left(-\pi Q^+_{z_0}(\mathbf{x},\mathbf{x})\right).$$

The form is now rapidly decreasing in $\mathbf{x}$, since the Siegel majorant

$$Q^+_{z_0}(\mathbf{x},\mathbf{x}) = \pi Q(\mathbf{x},\mathbf{x}) - 2\pi Q\big|_{z_0}(\mathbf{x},\mathbf{x})$$

is positive definite, thus

$$\varphi\big|_e \in \left[\bigwedge\nolimits^q \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q})\right]^{K_\infty}.$$

Theorem 3.2.5. *We have* $2^{-\frac{q}{2}}\varphi(\mathbf{x}) = \varphi_{KM}(\mathbf{x})$.

Proof. It is a straightforward computation to show that

$$(2\pi)^{-n_\alpha/2} H_{n_\alpha}\left(\sqrt{2\pi}x_\alpha\right)\exp(-\pi x_\alpha^2) = \left(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\right)^{n_\alpha}\exp(-\pi x_\alpha^2).$$

Hence applying this we find that $\varphi_{KM}\big|_e(\mathbf{x})$, defined by the Howe operators in (2.2.5), is

$$2^{-q}(2\pi)^{-\frac{q}{2}} \sum \omega_{\alpha_1 p+1} \wedge \cdots \wedge \omega_{\alpha_q p+q} \otimes H_{n_1}\left(\sqrt{2\pi}x_1\right) \cdots H_{n_p}\left(\sqrt{2\pi}x_p\right) \exp\left(-\pi Q\big|_{z_0}(\mathbf{x},\mathbf{x})\right).$$

Comparing with (3.2.3) we find that

$$\varphi_{KM}\big|_e(\mathbf{x}) = 2^{-\frac{q}{2}}e^{-\pi Q(\mathbf{x},\mathbf{x})}\varphi^0\big|_e(\mathbf{x}) = 2^{-\frac{q}{2}}\varphi\big|_e(\mathbf{x}).$$

□

3.3 Examples

1. Let us compute the Kudla-Millson as above in the simplest setting of signature $(1,1)$. Let $X_\mathbb{R} = \mathbb{R}^2$ with the quadratic form $Q(\mathbf{x},\mathbf{y}) = x'y + xy'$ where $\mathbf{x} = (x,x')$ and $\mathbf{y} = (y,y')$. Let $\mathbf{e}_1 = \frac{1}{\sqrt{2}}(1,1)$ and $\mathbf{e}_2 = \frac{1}{\sqrt{2}}(1,-1)$. We identify $z_0 = \mathbb{R}\mathbf{e}_2$ with $\mathbb{R}$, where the quadratic form is

simply $-r^2$. The projection map is given by

$$\begin{aligned} \mathrm{pr}\colon X_{\mathbb{R}} &\longrightarrow z_0 \\ \mathbf{x} = (x, x') &\longmapsto \frac{x - x'}{\sqrt{2}}. \end{aligned}$$

The orthogonal group of $X_{\mathbb{R}}$ is

$$H(\mathbb{R})^+ = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t > 0 \right\},$$

and we have $\mathbb{D}^+ = \mathbb{R}_{>0}$. The bundle is $E = \mathbb{R}_{>0} \times \mathbb{R}$, and the connection is $\nabla = d$ since the bundle is trivial. Hence the Mathai-Quillen form is

$$U_{MQ} = \sqrt{2} e^{-2\pi r^2} dr \in \Omega^1(E),$$

as in the proof of Proposition 3.1.5. For $\mathbf{x} = (x, x')$ the section $s_{\mathbf{x}} \colon \mathbb{R}_{>0} \to E$ is given by

$$s_{\mathbf{x}}(t) = \left(t, \frac{t^{-1}x - tx'}{\sqrt{2}} \right),$$

and we obtain

$$s_{\mathbf{x}}^* U_{MQ} = e^{-\pi \left(\frac{x}{t} - tx' \right)^2} \left(\frac{x}{t} + tx' \right) \frac{dt}{t}.$$

Hence after multiplication by $2^{-\frac{1}{2}} e^{-\pi Q(\mathbf{x},\mathbf{x})}$ we get

$$\varphi_{KM}(\mathbf{x}) = 2^{-\frac{1}{2}} e^{-\pi \left[\left(\frac{x}{t} \right)^2 + (tx')^2 \right]} \left(\frac{x}{t} + tx' \right) \frac{dt}{t}$$

2. The second example illustrates the functorial properties of the Mathai-Quillen form. Suppose that we have an orthogonal splitting $X_{\mathbb{R}} = \bigoplus_i^r X_{\mathbb{R}}^i$, where $X_{\mathbb{R}}^i$ has signature (p_i, q_i). Let $\mathbb{D}_i$ be the symmetric space associated to $X_{\mathbb{R}}^i$. We view the product $\mathbb{D}_1 \times \cdots \times \mathbb{D}_r$ as the subspace of $\mathbb{D}^+$:

$$\mathbb{D}_1 \times \cdots \times \mathbb{D}_r \simeq \left\{ z \in \mathbb{D} \mid z = \bigoplus_{i=1}^r z \cap X_{\mathbb{R}}^i \right\}.$$

Suppose we fix $z_0 = z_0^1 \oplus \cdots \oplus z_0^r \in \mathbb{D}_1^+ \times \cdots \times \mathbb{D}_r^+ \subset \mathbb{D}$, where z_0^i is a negative q_i-plane in $X_{\mathbb{R}}^i$. Let $H_{\mathbb{R}}^i$ be the subgroup preserving $X_{\mathbb{R}}^i$, let $K_{\infty,i}$ the stabilizer of z_0^i, and $\mathbb{D}_i$ be the symmetric space associated to $X_{\mathbb{R}}^i$.

Over $\mathbb{D}_1^+ \times \cdots \times \mathbb{D}_r^+$ the bundle E splits as an orthogonal sum $E = E_1 \oplus \cdots \oplus E_r$, where $E_i = H_i(\mathbb{R})^+ \times_{K_{\infty,i}} z_0^i$. Moreover the restriction of the Mathai-Quillen form to this subbundle is

$$U_{MQ}\big|_{E_1 \times \cdots \times E_r} = U_{MQ}^1 \wedge \cdots \wedge U_{MQ}^r,$$

where U_{MQ}^i is the Mathai-Quillen form on E_i. The section $s_{\mathbf{x}}$ also splits as a direct sum $\oplus s_{\mathbf{x}_i}$ where $\mathbf{x}_i$ is the projection of $\mathbf{x}$ onto $X_{\mathbb{R}}^i$. In summary the following diagram commutes

$$\begin{array}{ccc} E_1 \oplus \cdots \oplus E_r & \hookrightarrow & E \\ {\scriptstyle \oplus s_{\mathbf{x}_i}} \uparrow & & \uparrow {\scriptstyle s_{\mathbf{x}}} \\ \mathbb{D}_1^+ \times \cdots \times \mathbb{D}_r^+ & \hookrightarrow & \mathbb{D}^+ \end{array},$$

and we can conclude that

$$\varphi_{KM}(\mathbf{x})\big|_{\mathbb{D}_1^+ \times \cdots \times \mathbb{D}_r^+} = \varphi_{KM}^1(\mathbf{x}_1) \wedge \cdots \wedge \varphi_{KM}^r(\mathbf{x}_r)$$

where φ_{KM}^i is the Kudla-Millson form on $\mathbb{D}_i^+$.

Chapter 4

Diagonal restriction of Eisenstein series

In the previous chapter we considered the Kudla-Millson form for an arbitrary quadratic space. In this chapter, we specialize to a case where $X_{\mathbb{Q}}$ is obtained by restriction of scalars and has parallel signature (N, N). The space M_K is of dimension N^2, and the theta series $\Theta_{KM}(\tau, \varphi_f)$ is an N-form. Given a totally odd finite order unitary[1] Hecke character $\psi\colon F^\times\backslash\mathbb{A}_F^\times \longrightarrow U(1)$ we define in Section 4.3.2 a relative cycle

$$C \otimes \psi \in \mathcal{Z}_N(\overline{M_K}, \partial\overline{M_K}; \mathbb{R})$$

on which we will integrate $\Theta_{KM}(\tau, \varphi_f)$. We show that this integral is the diagonal restriction of an Eisenstein series and we compute the Fourier coefficients in terms of intersection numbers. In Section 4.4 we specialize to the case where F is a quadratic field and recover [DPV21, Theorem. A]; see Corollary 4.4.12.1.

4.1 Notations

4.1.1 Number fields

Let $F = \mathbb{Q}(\lambda)$ be a totally real field of degree N, let f_λ be the minimal polynomial of λ, and $\mathscr{O}$ be the ring of integers. We will write $F_{\mathbb{Q}}$ when we want to emphasize that we view F as a $\mathbb{Q}$-vector space. Let $\epsilon_1, \dots, \epsilon_N$ be an oriented $\mathbb{Z}$-basis of $\mathscr{O}$. With this basis we identify $F_{\mathbb{Q}} \simeq \mathbb{Q}^N$ as column vectors. For $\mu \in F^\times$ we define the map $\gamma(\mu) \in \mathrm{GL}_N(\mathbb{Q})$ obtained by left multiplication on $F_{\mathbb{Q}}$. The image of this representation is

$$C_\lambda(\mathbb{Q}) \coloneqq \{ g \in \mathrm{GL}_N(\mathbb{Q}) |\ g\gamma(\lambda) = \gamma(\lambda) g \},$$

the centralizer of $\gamma(\lambda)$. It is a maximal $\mathbb{R}$-split torus in $\mathrm{GL}_N(\mathbb{Q})$. We have a non-degenerate pairing

$$\begin{aligned} F \times F &\longrightarrow \mathbb{Q} \\ (x, y) &\longmapsto \mathrm{tr}_{F/\mathbb{Q}}(xy), \end{aligned} \tag{4.1.1}$$

[1] In this chapter ψ will always be unitary

and the dual of $\mathscr{O}$ is the inverse different ideal $\mathfrak{d}_F^{-1}$. Let $\sigma_1, \dots, \sigma_N$ be the N real embeddings of F. We order them such that the matrix

$$g_\infty \coloneqq \begin{pmatrix} \sigma_1(\epsilon_1) & \dots & \sigma_1(\epsilon_N) \\ \vdots & & \vdots \\ \sigma_N(\epsilon_1) & \dots & \sigma_N(\epsilon_N) \end{pmatrix} \in \mathrm{GL}_N(\mathbb{R}) \tag{4.1.2}$$

has positive determinant. We let

$$A \coloneqq {}^{\mathrm{t}}g_\infty g_\infty = (\mathrm{tr}_{F/\mathbb{Q}}(\epsilon_i \epsilon_j))_{ij} \in \mathrm{GL}_N(\mathbb{Q}), \tag{4.1.3}$$

whose determinant

$$d_F = \det(A)$$

is the discriminant of F.

4.1.2 Restriction of scalars

For any v we define the N-dimensional $\mathbb{Q}_v$-algebra $F_v \coloneqq \prod_{w|v} F_w$, and for a finite place $v = p$ we set $\mathscr{O}_p \coloneqq \prod_{w|p} \mathscr{O}_w$. At every place we have an embedding

$$\begin{aligned} F &\hookrightarrow F_w \\ \lambda &\longmapsto \lambda_w, \end{aligned}$$

in particular when v is totally split we have N such embeddings. Let us denote by $F_{\mathbb{Q}_v}$ the $\mathbb{Q}_v$-algebra $F_\mathbb{Q} \otimes \mathbb{Q}_v$. We have an isomorphism of $\mathbb{Q}_v$-algebras:

$$\begin{aligned} \varsigma_v \colon F_{\mathbb{Q}_v} &\longrightarrow F_v \\ \alpha \otimes t &\longmapsto (\alpha_w t)_{w|v}, \end{aligned} \tag{4.1.4}$$

which induces an isomorphism $\mathscr{O} \otimes_\mathbb{Z} \mathbb{Z}_p \simeq \mathscr{O}_p$ at the finite places; see [Wie85] for a proof. We fix a basis $\mathbb{Z}_v$ basis of $\mathscr{O}_w$ for $w \mid v$ which induces an isomorphism $F_v \simeq \mathbb{Q}_v^N$. For $t \in F_v^\times$ we let $g(t) \in \mathrm{GL}(F_{\mathbb{Q}_v}) \simeq \mathrm{GL}_N(\mathbb{Q}_v)$ be the map induced from left multiplication on F_v. Then there is $g_v \in \mathrm{Hom}(F_{\mathbb{Q}_v}, F_v) \simeq \mathrm{GL}_N(\mathbb{Q}_v)$ such that for every $t \in F$ we have

$$\gamma(\varsigma^{-1} t) = g_v^{-1} g(t) g_v \in \mathrm{GL}_N(\mathbb{Q}_v).$$

However we will usually identify $F_{\mathbb{Q}_v}$ and F_v and simply write $\gamma(t)$ instead of $\gamma(\varsigma^{-1}t)$. In the case where $v=\infty$, then $F_\infty = \mathbb{R}^N$. For $t=(t_1,\ldots,t_N)\in F_\infty$, we simply have $g(t) = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_N \end{pmatrix}$ and we can take g_∞ as in (4.1.2). Let $F_{\mathbb{A}} \coloneqq F_{\mathbb{Q}} \otimes \mathbb{A}$. The map $\varsigma \coloneqq \otimes_v \varsigma_v$ induces an isomorphism

$$\varsigma\colon F_{\mathbb{A}} \longrightarrow \mathbb{A}_F,$$

which also identifies $\widehat{\mathbb{Z}} \otimes \mathscr{O} \simeq \widehat{\mathscr{O}}$.

4.1.3 Zeta functions and L-functions

For a Schwartz function $\Phi \in \mathscr{S}(\mathbb{A}_F)$ we define the *Zeta function*

$$\zeta(\Phi,\psi,s) \coloneqq \int_{\mathbb{A}_F^\times} \Phi(x)\psi(x)|x|^s dt^\times.$$

This function converges absolutely for $\mathrm{Re}(s)>1$ by [Bum97, Proposition. 3.1.4 (iii)]. For a decomposable Schwartz function $\Phi = \otimes\Phi_w \in \mathscr{S}(\mathbb{A}_F)$ and for $\mathrm{Re}(s)>1$ the decomposition

$$\zeta(\Phi,\psi,s) = \prod_w \zeta_w(\Phi,\psi,s)$$

is valid, where the product is taken over the places of F and ζ_w are the local zeta integrals

$$\zeta_w(\Phi,\psi,s) \coloneqq \int_{F_w^\times} \Phi_w(t_w)\psi_w(t_w)|t_w|^s dt_w^\times$$

which converge for $\mathrm{Re}(s)>0$. We denote by ζ_f the finite part of the zeta integral:

$$\zeta_f(\Phi,\psi,s) \coloneqq \int_{\mathbb{A}_{F,f}^\times} \Phi_f(t_f)\psi_f(t_f)|t_f|^s dt_f^\times.$$

The zeta function $\zeta(\Phi,\psi,s)$ admits a meromorphic continuation to the complex plane and a functional equation

$$\zeta(\Phi,\psi,s) = \zeta(\Phi^\vee,\psi^{-1},1-s),$$

where $\Phi^\vee \in \mathscr{S}(\mathbb{A}_F)$ is the Fourier transform of Φ defined in (2.1.4). In our case, at an archimedean place σ, the Schwartz function will be $\Phi_\sigma(t) = e^{-\pi t^2} t$ and the character $\psi_\sigma(t) = \mathrm{sgn}(t)$. Then

$$\begin{aligned}\zeta_\infty(\Phi, \psi, s) &= 2^N \left(\prod_{\sigma|\infty} \int_0^\infty \Phi_\sigma(t_\sigma) t_\sigma^s \frac{dt_\sigma}{t_\sigma} \right) \\ &= 2^N \left(\prod_{\sigma|\infty} \int_0^\infty e^{-\pi v_\sigma t_\sigma^2} v_\sigma t_\sigma^{1+s} \frac{dt_\sigma}{t_\sigma} \right) \\ &= \Lambda(s).\end{aligned}$$

where

$$\Lambda(s) := \Gamma\left(\frac{1+s}{2}\right)^N \pi^{-\frac{N(1+s)}{2}} \tag{4.1.5}$$

At the places where $\Phi_w(x) = \mathbf{1}_{\mathscr{O}_w}(x)$ and the character ψ is unramified we have

$$\begin{aligned}\zeta_w(\Phi, \psi, s) &= \int_{\mathscr{O}_w^*} \psi_w(t_w) |t_w|^s dt_w^\times \\ &= \sum_{m=0}^{\infty} \int_{\mathfrak{p}_w^m} \psi_w(t_w) |t_w|^s dt_w^\times \\ &= \sum_{m=0}^{\infty} \int_{\mathscr{O}_w^\times} \psi_w(\pi_w)^m q_w^{-ms} \psi(t_w) dt_w^\times \\ &= \mathrm{vol}^\times(\mathscr{O}_w^\times) \frac{1}{1 - \psi_w(\pi_w) q_w^{-s}} \\ &= \mathrm{vol}^\times(\mathscr{O}_w^\times) L_w(\psi, s) \end{aligned} \tag{4.1.6}$$

where $\mathscr{O}_w^* = \mathscr{O}_w - \{0\}$ and $L_w(\psi, s)$ are the local factors as in Subsection 2.1.8 for $\mathfrak{p}$ corrsponding w.

4.1.4 Hilbert modular forms and Eisenstein series

Let $\Gamma \subset \mathrm{SL}_2(\mathscr{O})$ be a finite index subgroup. Let $(\tau_1, \ldots, \tau_N)$ be an element in $\mathbb{H}^N := \mathbb{H} \times \cdots \times \mathbb{H}$. The group Γ acts on $\mathbb{H}^N$ by

$$\gamma(\tau_1, \ldots, \tau_N) = (\gamma_1 \tau_N, \ldots, \gamma_N \tau_N),$$

where $\gamma_i = \sigma_i(\gamma) \in \mathrm{SL}_2(\mathbb{R})$. A holomorphic function $f \colon \mathbb{H}^N \longrightarrow \mathbb{C}$ is a Hilbert modular form of weight $(k_1, \ldots, k_N)$ for Γ if

1. $f(\gamma_1\tau_N, \dots, \gamma_N\tau_N) = (c_1\tau_1 + d_1)^{k_1} \cdots (c_N\tau_N + d_N)^{k_N} f(\tau_1, \dots, \tau_N)$, where $\gamma_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$,

2. if $F = \mathbb{Q}$ the f is holomorphic at the cusps.

We say that f has parallel weight k if $k_1 = \cdots = k_N = k$. If f is a Hilbert modular form then its diagonal restriction $f(\tau) \coloneqq f(\tau, \dots, \tau)$ is a modular form of weight $k_1 + \cdots + k_N$ for $\Gamma \cap \mathrm{SL}_2(\mathbb{Z})$.

Let $\phi_\sigma \in \mathscr{S}(F_\sigma^2)$ be defined by

$$\phi_\sigma(\mathbf{x}_\sigma) = (-i)e^{-\pi|z_\sigma|^2} z_\sigma$$

where $\mathbf{x}_\sigma = (x_\sigma, x'_\sigma)$ and $z_\sigma \coloneqq x_\sigma + i x'_\sigma$. Let $\phi = \phi_\infty \otimes \phi_f \in \mathscr{S}(F_{\mathbb{A}}^2)$ where $\phi_\infty = \prod_\sigma \phi_\sigma \in \mathscr{S}(F_\infty^2)$ and $\phi_f \in \mathscr{S}(F_{\mathbb{A}_f}^2)$ an arbitrary finite Schwartz function. For a totally odd finite order Hecke character ψ we define the function

$$Z(g, \mathbf{x}, \phi, \psi, s) \coloneqq \int_{\mathbb{A}_F^\times} \omega_l(g,t)\phi(\mathbf{x})\psi(t)|t|^s dt^\times,$$

which converges absolutely for $\mathrm{Re}(s) > 0$. For $\underline{\tau} = \underline{u} + i\underline{v} = (\tau_1, \dots, \tau_N) \in \mathbb{H}^N$ let

$$g_{\underline{\tau}} = \begin{pmatrix} \sqrt{\underline{v}} & \underline{u}/\sqrt{\underline{v}} \\ 0 & 1/\sqrt{\underline{v}} \end{pmatrix} \in \mathrm{SL}_2(F_\infty) \simeq \mathrm{SL}_2(\mathbb{R})^N$$

be the element that sends $(i, \dots, i)$ to $(\tau_1, \dots, \tau_N)$. We define the Eisenstein series

$$E(\tau_1, \dots, \tau_N, \phi_f, \psi, s) \coloneqq \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} (v_1 \cdots v_N)^{-\frac{1}{2}} Z\left(g_{\underline{\tau}}, \gamma_0^{-1}\mathbf{x}_0, \phi, \psi, s\right), \tag{4.1.7}$$

where $P(F)$ is the stabilizer of $\mathbf{x}_0 \coloneqq {}^{\mathrm{t}}(1,0)$ and $\gamma_0 \in \mathrm{GL}_2(F)$ is one of the following representatives for $P(F)\backslash \mathrm{GL}_2(F)$

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \text{ with } \lambda \in F^\times.$$

The Eisenstein series converges termwise absolutely for $\mathrm{Re}(s) > N - 1$, see [Wie85, Lemme p. 106]. It admits an analytic continuation to the whole plane by [Wie85, Proposition. 9] and we set

$$E(\tau_1, \dots, \tau_N, \phi_f, \psi) \coloneqq E(\tau_1, \dots, \tau_N, \phi_f, \psi, s)\big|_{s=0}.$$

Since the Schwartz function decomposes as $\phi = \phi_\infty \otimes \phi_f$ we can decompose the integral

$$Z\left(g_{\underline{\tau}}, \begin{pmatrix} m \\ n \end{pmatrix}, \phi, \psi, s\right) = Z_\infty\left(g_{\underline{\tau}}, \begin{pmatrix} m \\ n \end{pmatrix}, \phi, \psi, s\right) Z_f\left(1, \begin{pmatrix} m \\ n \end{pmatrix}, \phi, \psi, s\right).$$

Lemma 4.1.1. *We have*

$$Z_\infty\left(g_{\underline{\tau}}, \begin{pmatrix} m \\ n \end{pmatrix}, \phi, \psi, s\right) = \frac{\Lambda(1+s)}{(i\pi)^N} \frac{(v_1 \cdots v_N)^{\frac{1}{2}+s}}{\mathrm{N}(m - n\underline{\tau})|\,\mathrm{N}(m - n\underline{\tau})|^s}.$$

Proof. Let $\tau_\sigma = u_\sigma + iv_\sigma$ and $g_{\tau_\sigma} = \begin{pmatrix} \sqrt{v_\sigma} & u_\sigma/\sqrt{v_\sigma} \\ 0 & 1/\sqrt{v_\sigma} \end{pmatrix}$, then

$$g_{\tau_\sigma}^{-1} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} \alpha_\sigma \\ \beta_\sigma \end{pmatrix}$$

with $\alpha_\sigma = \frac{m - nu_\sigma}{\sqrt{v_\sigma}}$ and $\beta_\sigma = n\sqrt{v_\sigma}$. Thus

$$\int_{F_\infty^\times} \omega_l(g_{\underline{\tau}})\phi_\infty \begin{pmatrix} mt_\infty \\ nt_\infty \end{pmatrix} \psi_\infty(t_\infty)|t_\infty|^{1+s} dt_\infty^\times = 2^N \prod_\sigma \int_0^\infty \phi_\sigma \begin{pmatrix} \alpha_\sigma t_\sigma \\ \beta_\sigma t_\sigma \end{pmatrix} t_\sigma^{1+s} \frac{dt_\sigma}{t_\sigma},$$

since ϕ_σ and ψ_σ are both odd functions. At the place σ we have

$$\int_0^\infty \phi_\sigma \begin{pmatrix} \alpha_\sigma t_\sigma \\ \beta_\sigma t_\sigma \end{pmatrix} t_\sigma^{1+s} \frac{dt_\sigma}{t_\sigma} = -iz_\sigma \int_0^\infty e^{-\pi t_\sigma^2 |z_\sigma|^2} t_\sigma^{2+s} \frac{dt_\sigma}{t_\sigma},$$

where $z_\sigma = \alpha_\sigma + i\beta_\sigma = \overline{\frac{m - n\tau_\sigma}{\sqrt{v_\sigma}}}$. The right hand side converges for $\mathrm{Re}(s) > -2$ to

$$-\frac{iz_\sigma}{2|z_\sigma|^{2+s}} \Gamma\left(1 + \frac{s}{2}\right) \pi^{-\left(1+\frac{s}{2}\right)} = \frac{1}{2i\pi} \Gamma\left(1 + \frac{s}{2}\right) \pi^{-\frac{s}{2}} \frac{v_\sigma^{\frac{1}{2}+s}}{(m - n\tau_\sigma)|m - n\tau_\sigma|^s}.$$

Hence

$$Z_\infty\left(g_{\underline{\tau}}, \begin{pmatrix} m \\ n \end{pmatrix}, \phi, \psi, s\right) = \frac{\Lambda(1+s)}{(i\pi)^N} \frac{(v_1 \cdots v_N)^{\frac{1}{2}+s}}{\mathrm{N}(m - n\underline{\tau})|\,\mathrm{N}(m - n\underline{\tau})|^s}.$$

□

Proposition 4.1.2. *For* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F)^+$ *and* ϕ_f *arbitrary we have*

$$E(\gamma\underline{\tau}, \phi_f, \psi, s) = |\det(\gamma)|^{\frac{1}{2}}\, \mathrm{N}(c\underline{\tau} + d)|\,\mathrm{N}(c\underline{\tau} + d)|^s E(\underline{\tau}, \omega_l(\gamma^{-1})\phi_f, \psi, s).$$

In particular $E(\underline{\tau}, \phi_f, \psi)$ is a Hilbert modular form of parallel weight one for any $\Gamma \subset \mathrm{SL}_2(\mathscr{O})$ that preserves ϕ_f.

Proof. By the GL_2-invariance of the sum in (4.1.7) we have

$$\sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} Z\left(g_{\underline{\tau}}, \gamma_0^{-1}\mathbf{x}_0, \phi, \psi, s\right) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} Z\left(g_{\underline{\tau}}, \gamma\gamma_0^{-1}\mathbf{x}_0, \phi, \psi, s\right).$$

Hence $E(\underline{\tau}, \omega_l(\gamma^{-1})\phi_f, \psi, s)$ is equal to

$$\begin{aligned}
&(v_1 \cdots v_N)^{-\frac{1}{2}} \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} Z_\infty\left(g_{\underline{\tau}}, \gamma_0^{-1}\mathbf{x}_0, \phi_\infty, \psi_\infty, s\right) Z_f\left(1, \gamma_0^{-1}\mathbf{x}_0, \omega_l(\gamma^{-1})\phi_f, \psi_f, s\right) \\
&= |\det(\gamma)|^{-\frac{1}{2}} (v_1 \cdots v_N)^{-\frac{1}{2}} \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} Z_\infty\left(g_{\underline{\tau}}, \mathbf{x}_0, \phi_\infty, \psi_\infty, s\right) Z_f\left(1, \gamma\gamma_0^{-1}\mathbf{x}_0, \phi_f, \psi_f, s\right) \\
&= |\det(\gamma)|^{-\frac{1}{2}} (v_1 \cdots v_N)^{-\frac{1}{2}} \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} Z_\infty\left(g_{\underline{\tau}}, \gamma^{-1}\gamma_0^{-1}\mathbf{x}_0, \phi_\infty, \psi_\infty, s\right) Z_f\left(1, \gamma_0^{-1}\mathbf{x}_0, \phi_f, \psi_f, s\right)
\end{aligned} \tag{4.1.8}$$

In Lemma 4.1.1, we see that replacing $\begin{pmatrix} m \\ n \end{pmatrix}$ by $\gamma^{-1}\begin{pmatrix} m \\ n \end{pmatrix}$ gives

$$Z_\infty\left(g_{\underline{\tau}}, \gamma^{-1}\begin{pmatrix} m \\ n \end{pmatrix}, \phi, \psi, s\right) = \frac{1}{\mathrm{N}(c\underline{\tau}+d)|\,\mathrm{N}(c\underline{\tau}+d)|^s} Z_\infty\left(g_{\gamma\underline{\tau}}, \begin{pmatrix} m \\ n \end{pmatrix}, \phi, \psi, s\right),$$

and the righthandside of (4.1.8) is

$$|\det(\gamma)|^{-\frac{1}{2}}\,\mathrm{N}(c\underline{\tau}+d)^{-1}|\,\mathrm{N}(c\underline{\tau}+d)|^{-s} E(\gamma\underline{\tau}, \phi_f, \psi, s).$$

□

4.2 The setting

4.2.1 The quadratic space

Let F be a totally real field of degree N with ring of integers $\mathscr{O}$. Let $X_F^0 := F^2$ be the 2-dimensional quadratic F-space with the quadratic form $Q^0(\mathbf{x}, \mathbf{y}) = xy' + x'y$ for $\mathbf{x} = (x, x')$ and $\mathbf{y} = (y, y')$ in F^2. It is represented by the symmetric matrix $A(Q^0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let $\mathrm{SO}(F^2)$ be its orthogonal group over F. We have an embedding

$$\begin{aligned} F^\times &\hookrightarrow \mathrm{SO}(F^2) \\ t &\longmapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}. \end{aligned} \tag{4.2.1}$$

The restriction of scalars $X_{\mathbb{Q}} := \mathrm{Res}_{F/\mathbb{Q}} F^2 \simeq F^2_{\mathbb{Q}}$ with $Q := \mathrm{tr}_{F/\mathbb{Q}} \circ Q^0$ is a $2N$ dimensional space over $\mathbb{Q}$. At every place v of $\mathbb{Q}$ let $X_{\mathbb{Q}_v} := X_{\mathbb{Q}} \otimes \mathbb{Q}_v = F^2_{\mathbb{Q}_v}$. Using the $\mathbb{Z}$-basis of $\mathscr{O}$ that we fixed we obtain an isomorphism $F^2_{\mathbb{Q}_v} \simeq \mathbb{Q}_v^{2N}$, equipped with the quadratic form

$$A(Q) = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$$

where $A = {}^{\mathrm{t}}g_\infty g_\infty$ is as in (4.1.3) and is positive definite. Let $H(\mathbb{Q}) = \mathrm{SO}(F^2_{\mathbb{Q}})$ be the orthogonal group of $X_{\mathbb{Q}}$, given by

$$H(\mathbb{Q}) = \left\{ h \in \mathrm{SL}_{2N}(\mathbb{Q}) \,\middle|\, {}^{\mathrm{t}}h \begin{pmatrix} & A \\ A & \end{pmatrix} h = \begin{pmatrix} & A \\ A & \end{pmatrix} \right\}.$$

4.2.2 The space M_K

The real vector space $X_{\mathbb{R}} = F^2_{\mathbb{R}}$ is of signature (N, N). Let $\mathbb{D}$ be the corresponding symmetric space. Consider the basis $\mathbf{e}_1, \dots, \mathbf{e}_N, \mathbf{f}_1, \dots, \mathbf{f}_N$ of $F^2_{\mathbb{R}} \simeq \mathbb{R}^{2N}$ where $\mathbf{e}_k := (e_k, e_k)$ and $\mathbf{f}_k := (e_k, -e_k)$ are in $F^2_{\mathbb{R}}$, and e_k is the standard unit vector in $\mathbb{R}^N$. We have $Q(\mathbf{e}_k, \mathbf{e}_k) = 2$ and $Q(\mathbf{f}_k, \mathbf{f}_k) = -2$. We orient $F^2_{\mathbb{R}}$ by

$$o(F^2_{\mathbb{R}}) := \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_N \wedge \mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_N.$$

As a basepoint we take the negative N-plane

$$z_0 := \left\{ \mathbf{x} = (v, -v) \in F^2_{\mathbb{R}} \,\middle|\, v \in F_{\mathbb{R}} \right\}$$

spanned by the vectors $\mathbf{f}_1, \dots, \mathbf{f}_N$. We fix the orientation $o(z_0) = \mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_N$. Let $\mathbb{D}^+$ be the connected component of $\mathbb{D}$ containing the oriented plane z_0.

Lemma 4.2.1. *For z_0 as above we have*

$$K_\infty(z_0) = h_\infty^{-1} \left\{ \frac{1}{2} \begin{pmatrix} k_1 + k_2 & k_1 - k_2 \\ k_1 - k_2 & k_1 + k_2 \end{pmatrix}, k_i \in \mathrm{SO}(N) \right\} h_\infty,$$

where $h_\infty := \begin{pmatrix} g_\infty & 0 \\ 0 & g_\infty \end{pmatrix}$.

Proof. We have an isomorphism of quadratic spaces

$$F_{\mathbb{R}}^2 \longrightarrow \mathbb{R}^{2N}$$
$$\begin{pmatrix} v \\ v' \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} v+v' \\ v-v' \end{pmatrix},$$

where the righthandside is equipped with the quadratic form ${}^{\mathrm{t}}xAx - {}^{\mathrm{t}}yAy$. It maps z_0 to the negative plane $x = 0$, whose stabilizer are the matrices $\begin{pmatrix} k_1' & 0 \\ 0 & k_2' \end{pmatrix}$ with ${}^{\mathrm{t}}k_i'Ak_i' = A$ *i.e.* $k_i' \in g_\infty^{-1}\,\mathrm{SO}(N)g_\infty$. Hence

$$K_\infty(z_0) = \left\{ \frac{1}{2}\begin{pmatrix} k_1'+k_2' & k_1'-k_2' \\ k_1'-k_2' & k_1'+k_2' \end{pmatrix}, k_i' \in g_\infty^{-1}\,\mathrm{SO}(N)g_\infty \right\}$$

and the result follows after setting $k_i' = g_\infty^{-1}k_i g_\infty$. □

We take a Schwartz function $\varphi_f \in \mathscr{S}(F_{\mathbb{A}}^2)$ fixed by a open compact $K_f \subset H(\mathbb{A}_f)$. Let $K = K_\infty K_f$ and define the space

$$M_K := H(\mathbb{Q})\backslash H(\mathbb{A})/K \simeq \bigsqcup_{i=1}^{r} M_{h_i}$$

where $M_{h_i} = \Gamma_{h_i}\backslash\mathbb{D}^+$, where $h_i \in H(\mathbb{A}_f)$ and Γ_{h_i} as in Subsection 2.2.5. Let

$$\Theta_{KM}(\tau, \varphi_f) \in \Omega^N(M_K)$$

be the Kudla-Millson theta series, defined in Subsection 2.2.9.

4.3 Integral of Θ_{KM} over a relative class $C \otimes \psi$

We will now define a relative cycle $C\otimes\psi \in \mathcal{Z}_N(\overline{M_K}, \partial\overline{M_K}; R)$ and compute the pairing $\int_{C\otimes\psi}\Theta_{KM}(\tau,\varphi_f)$.

4.3.1 The seesaw

Let $W_F^0 = X_F^0 \oplus X_F^0$ be the 4-dimensional symplectic F-space as in Subsection 2.1.9, and let $W_{\mathbb{Q}} = \mathrm{Res}_{F/\mathbb{Q}} W_F^0$ be the restriction of scalars. Then $W_{\mathbb{Q}} = X_{\mathbb{Q}} \oplus X_{\mathbb{Q}}$ and it is a $4N$-dimensional symplectic $\mathbb{Q}$-space. Let F_w^2 be the completion of the quadratic space at a place w of F. The isomorphism $\varsigma_v \colon F_{\mathbb{Q}_v} \longrightarrow F_v$ from (4.1.4) induces an isomorphism of quadratic $\mathbb{Q}_v$-spaces

$$F_v^2 \coloneqq \bigoplus_{w|v} F_w^2 \simeq F_{\mathbb{Q}_v}^2.$$

Hence we obtain a natural embedding $\mathrm{SO}(F_v^2) \subset H(\mathbb{Q}_v)$. We compose this embedding with the isomorphism (4.2.1) to get

$$h \colon F_v^\times \hookrightarrow H(\mathbb{Q}_v),$$

that we will describe more concretely. First note that we can embed $\mathrm{GL}_N(\mathbb{Q}_v) \subset H(\mathbb{Q}_v)$ by

$$\begin{aligned} \mathrm{GL}_N(\mathbb{Q}_v) &\hookrightarrow H(\mathbb{Q}_v) \\ M &\longmapsto \begin{pmatrix} {}^{\#}M & 0 \\ 0 & M \end{pmatrix} \end{aligned}$$

where ${}^{\#}M = A^{-1}\,{}^{\mathrm{t}}M^{-1}A$. The embedding h is obtained by composing it with the embedding $\gamma \colon F_v^\times \hookrightarrow \mathrm{GL}_N(\mathbb{Q}_v)$:

$$\begin{aligned} h \colon F_v^\times &\hookrightarrow H(\mathbb{Q}_v) \\ t_v &\longmapsto \begin{pmatrix} {}^{\#}\gamma(t_v) & 0 \\ & \\ 0 & \gamma(t_v) \end{pmatrix}. \end{aligned}$$

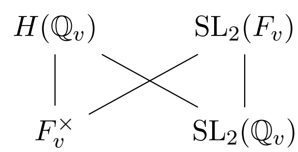
Note that at infinity we have $\gamma(t_\infty) = g_\infty^{-1} g(t_\infty) g_\infty$, hence ${}^{\#}\gamma(t_\infty) = \gamma(t_\infty^{-1})$ and

$$h(t_\infty) = h_\infty^{-1} \begin{pmatrix} g(t_\infty)^{-1} & 0 \\ & \\ 0 & g(t_\infty) \end{pmatrix} h_\infty$$

where $h_\infty = \begin{pmatrix} g_\infty & 0 \\ 0 & g_\infty \end{pmatrix}$. The centralizer of $F_v^\times$ in $\mathrm{Sp}(W_{\mathbb{Q}_v})$ is $\mathrm{SL}_2(F_v) = \prod \mathrm{SL}_2(F_w)$, embedded in $\mathrm{Sp}(W_{\mathbb{Q}_v})$ by

$$\begin{aligned} \mathrm{SL}_2(F_v) &\hookrightarrow \mathrm{Sp}(W_{\mathbb{Q}_v}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} \gamma(a) & & \gamma(b) & \\ & \gamma(a) & & \gamma(b) \\ \gamma(c) & & \gamma(d) & \\ & \gamma(c) & & \gamma(d) \end{pmatrix}. \end{aligned}$$

We obtain a dual pair $F_v^\times \times \mathrm{SL}_2(F_v) \subset \mathrm{Sp}(W_{\mathbb{Q}_v})$ and a seesaw



where the righthand side is the diagonal embedding

$$\iota_\Delta \colon \mathrm{SL}_2(\mathbb{Q}_v) \longrightarrow \mathrm{SL}_2(F_v).$$

4.3.2 The twisted class $C \otimes \psi$

The space F_σ^2 is of signature $(1,1)$. Let $\mathbb{D}_\sigma \simeq \mathbb{R}^\times$ be the corresponding symmetric space. We have an isomorphism $F_\mathbb{R}^2 \simeq F_\infty^2 = \oplus_\sigma F_\sigma^2$ of quadratic spaces. We can view the product of symmetric spaces $\mathbb{D}_0 \coloneqq \prod \mathbb{D}_\sigma$ as the subspace

$$\mathbb{D}_0 = \mathbb{D}_{\sigma_1} \times \cdots \times \mathbb{D}_{\sigma_N} \simeq \left\{ z \in \mathbb{D} \mid z = \oplus_\sigma (z \cap F_\sigma^2) \right\},$$

of $\mathbb{D}$, and $\mathbb{D}_0^+ \simeq \mathbb{R}_{>0}^N$. The image of $F_\infty^\times = \prod F_\sigma^\times$ by h is precisely the stabilizer of this subspace in $\mathbb{D}$.

Let $\psi \colon F^\times \backslash \mathbb{A}_F^\times \longrightarrow U(1)$ be a totally odd Hecke character of finite order and conductor $\mathfrak{f}$. Let K_∞^0 be the maximal compact connected subgroup

$$K_\infty^0 \coloneqq \{ t_\infty \in \{\pm 1\}^N, \det(t_\infty) = 1 \} \subset F_\infty^\times.$$

We set

$$M_\mathfrak{f} \coloneqq F^\times \backslash \mathbb{A}_F^\times / K^0(\mathfrak{f}) \simeq F^\times \backslash \mathbb{D}_0^+ \times \mathbb{A}_{F,f}^\times / \widehat{U}(\mathfrak{f})$$

where $K^0(\mathfrak{f}) = K_\infty^0 \times \widehat{U}(\mathfrak{f})$. Suppose that $\mathfrak{f} \subset \mathscr{O}$ is small enough that $h(\widehat{U}(\mathfrak{f})) \subset K_f$. Then the embedding h induces an immersion

$$h \colon M_\mathfrak{f} \longrightarrow M_K. \tag{4.3.1}$$

We have a decomposition

$$\mathbb{A}_{F,f}^\times = \bigsqcup_{[\mathfrak{a}] \in \mathrm{Cl}_\mathfrak{f}(F)^+} F^{\times,+} t_\mathfrak{a} \widehat{U}(\mathfrak{f}). \tag{4.3.2}$$

The double coset space $M_{\mathfrak{f}}$ is the disjoint union of symmetric space

$$M_{\mathfrak{f}} = \bigsqcup_{[\mathfrak{a}] \in \mathrm{Cl}_{\mathfrak{f}}(F)^+} \Gamma_{\mathfrak{a}} \backslash \mathbb{D}_0^+,$$

where $\mathbb{D}_0^+ = F_\infty^{\times,+}$ and $\Gamma_{\mathfrak{a}} := F^{\times,+} \cap t_{\mathfrak{a}} \widehat{U}(\mathfrak{f}) t_{\mathfrak{a}}^{-1} = F^{\times,+} \cap \widehat{U}(\mathfrak{f})$, since $\mathbb{A}_F^\times$ is commutative. For every class $[\mathfrak{a}]$ in the ray class group let $C_{\mathfrak{a}} \in \mathcal{Z}_N(\overline{M_K}, \partial\overline{M_K}; \mathbb{R})$ be the image of the connected component $\Gamma_{\mathfrak{a}} \backslash F_\infty^{\times,+}$ by the map (4.3.1). It is non compact since

$$\Gamma_{\mathfrak{a}} \backslash F_\infty^{\times,+} \simeq \mathbb{R}_{>0} \times \Gamma_{\mathfrak{a}} \backslash F_\infty^{1,+},$$

where F^1 are the elements of norm 1. More precisely the map goes as follows: a point $\Gamma_{\mathfrak{a}} t_\infty$ is mapped to $F^\times(t_\infty, t_{\mathfrak{a}})K^0$, which is mapped to

$$H(\mathbb{Q})(h(t_\infty), h(t_{\mathfrak{a}}))K_f = H(\mathbb{Q})(z, h(t_{\mathfrak{a}}))K_f,$$

where $z = h(t_\infty)z_0 \in \mathbb{D}^+$. We write $h(t_{\mathfrak{a}}) = h_{\mathfrak{a}}^{-1} h_i K_f \in H(\mathbb{Q})^+ h_i K_f$, then

$$H(\mathbb{Q})(z, h(t_{\mathfrak{a}}))K_f = H(\mathbb{Q})(h_{\mathfrak{a}} z, h_i)K_f$$

is sent to $\Gamma_{h_i} h_{\mathfrak{a}} z$. If we set $\mathbb{D}_{\mathfrak{a}}^+ := h_{\mathfrak{a}} \mathbb{D}_0^+$ then $C_{\mathfrak{a}}$ is the image of $\mathbb{D}_{\mathfrak{a}}^+ \subset \mathbb{D}^+$ in M_{h_i}. We define the relative cycle

$$C \otimes \psi := \sum_{\mathfrak{a} \in \mathrm{Cl}_{\mathfrak{f}}(F)^+} \psi(\mathfrak{a}) C_{\mathfrak{a}} \in \mathcal{Z}_N(\overline{M_K}, \partial\overline{M_K}; \mathbb{R}),$$

where we view ψ as character on the ray class group.

4.3.3 Restriction to $\mathbb{D}_0^+$

Let $dt^\times$ be the Haar measure on $\mathbb{A}_F^\times$ normalized such that $\mathrm{vol}^\times(\widehat{U}(\mathfrak{f})) = 1$. We orient $F_\infty^{\times,+} \simeq \mathbb{R}_{>0}^N$ by the volume form $dt_\infty^\times = dt_1^\times \cdots dt_N^\times$ on $F_\infty^{\times,+} \simeq \mathbb{R}_{>0}^N$. This induces an isomorphism $\bigwedge^N \mathfrak{g}_0^* \simeq \mathbb{R}$ and

$$\Omega^N(\mathbb{D}_0^+) \simeq \left[C^\infty(\mathbb{R}_{>0}^N) \otimes \bigwedge^N \mathfrak{g}_0^* \right]^{K_\infty^0} \simeq C^\infty(\mathbb{R}_{>0}^N)^{K_\infty^0}$$

where $\mathfrak{g}_0 \simeq \mathbb{R}^N$ is the Lie algebra of $F_\infty^\times$ and the second map is given by evaluation on an oriented basis. Moreover, combining with (2.2.6) we get isomorphism

$$\begin{aligned} \left[C^\infty(F^\times\backslash\mathbb{A}_F^\times)\right]^{K^0(\mathfrak{f})} &\longrightarrow \Omega^N(M_\mathfrak{f}) \\ \eta_\infty \otimes \eta_f &\longmapsto \sum_{\mathfrak{a}\in \mathrm{Cl}_\mathfrak{f}(F)^+} \eta_f(t_\mathfrak{a})\eta_\infty dt_\infty^\times. \end{aligned} \tag{4.3.3}$$

If $\tilde{\eta}$ in $C^\infty(F^\times\backslash A_F^\times)^{K^0(\mathfrak{f})}$ corresponds to $\eta \in \Omega^N(M_\mathfrak{f})$ then

$$\int_{M_\mathfrak{f}} \eta = \frac{1}{\mathrm{vol}^\times(K_\infty^0)} \int_{F^\times\backslash\mathbb{A}_F^\times} \tilde{\eta}(t)dt^\times.$$

Since $K_\infty^0 \simeq \{\pm 1\}^{N-1}$ we have $\mathrm{vol}^\times(K_\infty^0) = 2^{N-1}$.

Recall that we have isomorphism

$$F_\mathbb{R}^2 \simeq F_\infty^2 = \prod_\sigma F_\sigma^2.$$

Let $(\mathbf{x}_{\sigma_1}, \ldots, \mathbf{x}_{\sigma_N}) \in F_\infty^2$ be the image of $\mathbf{x} = (x, x')$ in $F_\mathbb{R}^2$, where $\mathbf{x}_\sigma = (x_\sigma, x'_\sigma)$. We identify $F_\mathbb{R}$ with z_0 by sending v to $(v, -v)$, and consider the tautological bundle $E = H(\mathbb{R})^+ \times_{K_\infty} F_\mathbb{R}$ over $\mathbb{D}^+$, as in Chapter 3. The bundle splits over $\mathbb{D}_0^+$ *i.e.* we have the diagram

$$\begin{array}{ccc} E\big|_{\mathbb{D}_0^+} \simeq \mathbb{R}_{>0}^N \times F_\infty & \hookrightarrow & E \\ {\scriptstyle \oplus s_{\mathbf{x}_\sigma}}\uparrow & & \uparrow{\scriptstyle s_\mathbf{x}} \\ \mathbb{D}_0^+ \simeq \mathbb{R}_{>0}^N & \hookrightarrow & \mathbb{D}^+ \end{array}, \tag{4.3.4}$$

where the top map is given by

$$\begin{aligned} \mathbb{R}_{>0}^N &\longrightarrow E \\ (t_\infty, v) &\longmapsto [h(t_\infty), g_\infty^{-1} v]. \end{aligned}$$

Moreover the restriction of the section to $E\big|_{\mathbb{D}_0^+}$ is given by $\oplus_\sigma s_{\mathbf{x}_\sigma}$ where

$$s_{\mathbf{x}_\sigma}(t_\sigma) = \left(t_\sigma, \frac{t_\sigma^{-1}x_\sigma - t_\sigma x'_\sigma}{\sqrt{2}}\right) \in \mathbb{R}_{>0} \times F_\sigma.$$

Since $\mathbb{D}_\mathbf{x}^+$ is the zero locus $s_\mathbf{x}$ in $\mathbb{D}^+$, the intersection $\mathbb{D}_0^+ \cap \mathbb{D}_\mathbf{x}^+$ is the zero locus of $\oplus s_{\mathbf{x}_\sigma}$ in $\mathbb{D}_0^+$.

Proposition 4.3.1. *For $\mathbf{x} \in F_{\mathbb{R}}^2$ the restriction $\varphi_{KM}(\mathbf{x})\big|_{\mathbb{D}_0^+}$ corresponds to*

$$\omega_{os}(h(t_\infty))\varphi_\infty(\mathbf{x}) \in C^\infty(\mathbb{R}_{>0}^N)^{K_\infty^0}$$

where $\varphi_\infty(\mathbf{x}) = \prod_\sigma \varphi_\sigma(\mathbf{x}_\sigma) \in \mathscr{S}(F_\infty^2)$ and

$$\varphi_\sigma(\mathbf{x}_\sigma) = \exp\left(-\pi x_\sigma^2 - \pi {x'_\sigma}^2\right)(x_\sigma + x'_\sigma) \in \mathscr{S}(F_\sigma^2).$$

Proof. It follows from the two examples in Section 3.3 of Chapter 3. First we have

$$\varphi_{KM}(\mathbf{x})\big|_{\mathbb{D}_0^+} = \varphi_{KM}^{\sigma_1}(\mathbf{x}_{\sigma_1}) \wedge \cdots \wedge \varphi_{KM}^{\sigma_N}(\mathbf{x}_{\sigma_N})$$

where φ_{KM}^σ is the Kudla-Millson form of the symmetric space $\mathbb{D}_\sigma$ and $\mathbf{x}_\sigma$ is the component of $\mathbf{x}$ in F_σ^2. Secondly we have

$$\varphi_{KM}^\sigma(\mathbf{x}_{\sigma_1}) = \exp\left(-\pi\left(\frac{x_\sigma}{t_\sigma}\right)^2 - \pi\left(x'_\sigma t_\sigma\right)^2\right)\left(\frac{x_\sigma}{t_\sigma} + x'_\sigma t_\sigma\right)\frac{dt_\sigma}{t_\sigma}.$$

Note again that the Kudla-Millson form used here differs by a factor $2^{\frac{N}{2}}$ with the classical Kudla-Millson that appears in the original paper [KM90]. ☐

Let $\varphi_f \in \mathscr{S}(F_{\mathbb{A}_f}^2)$ be the K_f-invariant Schwartz function as before, that we view in $\mathscr{S}(\mathbb{A}_{F,f}^2) \simeq \mathscr{S}(F_{\mathbb{A}_f}^2)$. For $g \in \mathrm{SL}_2(\mathbb{A}_F)$ and $t \in \mathbb{A}_F^\times$ let us define

$$\widetilde{\Theta}_{os}(g,t,\varphi) \coloneqq \sum_{\mathbf{x}\in F^2} \omega_{os}(g,t)\varphi(\mathbf{x})$$

where $\varphi \coloneqq \varphi_\infty \otimes \varphi_f \in \mathscr{S}(\mathbb{A}_F^2)$; it is the kernel for the pair $\mathrm{SL}_2(\mathbb{A}_F) \times \mathbb{A}_F^\times$. We view $\mathbb{A}_F^\times \subset \mathrm{SO}(\mathbb{A}_F^2)$, so it is the ortho-symplectic pair given in Subsection 2.1.9. After fixing g we have a smooth function

$$\widetilde{\Theta}_{os}(g,\cdot,\varphi) \in \left[C^\infty(F^\times\backslash\mathbb{A}_F^\times)\right]^{K^0(\mathfrak{f})}$$

in the variable $t \in \mathbb{A}_F^\times$. Let

$$\Theta_{KM}(\tau,\varphi_f)\big|_{M_\mathfrak{f}} \in \Omega^N(M_\mathfrak{f})$$

be the restriction of the Kudla-Millson theta series to $M_\mathfrak{f}$. Let $g_\tau \in \mathrm{SL}_2(\mathbb{R})$ be the standard matrix sending i to τ.

Proposition 4.3.2. *The function $\widetilde{\Theta}_{os}(\iota_\Delta(g_\tau),\cdot,\varphi_f)$ correspond to $v^{\frac{N}{2}}\Theta_{KM}(\tau,\varphi_f)\big|_{M_\mathfrak{f}}$ in the isomorphism (4.3.3).*

Proof. Recall that we defined

$$\Theta_{KM}(\tau, \varphi_f) = v^{-\frac{N}{2}} \Theta_{os}(g_\tau, \varphi_{KM} \otimes \varphi_f).$$

First for $g \in \mathrm{SL}_2(\mathbb{A})$ we have

$$\begin{aligned}\Theta_{os}(g, \varphi_{KM} \otimes \varphi_f)\big|_{M_\mathfrak{f}} &= \Theta_{os}(g, \varphi_{KM}\big|_{\mathbb{D}_0^+} \otimes \varphi_f) \\ &= \sum_{\mathfrak{a} \in \mathrm{Cl}_\mathfrak{f}(F)^+} \Theta_{os}(g, h(t_\mathfrak{a}), \varphi_{KM}\big|_{\mathbb{D}_0^+} \otimes \varphi_f) \in \Omega^N(M_\mathfrak{f}).\end{aligned}$$

Then, using Proposition 4.3.1 we compute

$$\begin{aligned}\Theta_{os}(g, h(t_\mathfrak{a}), \varphi_{KM}\big|_{\mathbb{D}_0^+} \otimes \varphi_f) &= \sum_{\mathbf{x} \in F_\mathbb{Q}^2} \Big(\omega_{os}(g, t_\mathfrak{a})\varphi_{KM}\big|_{\mathbb{D}_0^+} \otimes \varphi_f\Big)(\mathbf{x}) \\ &= \sum_{\mathbf{x} \in F_\mathbb{Q}^2} \Big(\omega_{os}(g, h(t_\infty, t_\mathfrak{a}))\varphi_\infty \otimes \varphi_f\Big)(\mathbf{x}) dt_\infty^\times \\ &\overset{\text{seesaw}}{=} \sum_{\mathbf{x} \in F^2} \Big(\omega_{os}(\iota_\Delta(g), (t_\infty, t_\mathfrak{a}))\varphi_\infty \otimes \varphi_f\Big)(\mathbf{x}) dt_\infty^\times.\end{aligned}$$

At $g = (g_\tau, 1)$ and $t = (t_\infty, t_\mathfrak{a})$ it corresponds to $\widetilde{\Theta}_{os}(\iota_\Delta(g_\tau), t, \varphi)$ in the isomorphism (4.3.3). □

4.3.4 A few integrals

Before passing to the regularized integral of $\Theta_{KM}(\tau, \varphi_f)$ on $C \otimes \psi$, we will need the following integrals for $\mathbf{x} \in F_\mathbb{R}^2$ and $s \in \mathbb{C}$:

$$\begin{aligned}J_\infty(\mathbf{x}, s) &:= \prod_\sigma J_\sigma(\mathbf{x}_\sigma, s), \qquad J_\sigma(\mathbf{x}_\sigma, s) := \int_{\mathbb{R}^\times} \omega_{os}(t_\sigma)\varphi_\sigma(\mathbf{x}_\sigma)\psi(t_\sigma)|t_\sigma|^s dt_\sigma^\times, \\ J_\infty(\mathbf{x}, s)^+ &:= \prod_\sigma J_\sigma(\mathbf{x}_\sigma, s)^+, \qquad J_\sigma(\mathbf{x}_\sigma, s)^+ := \int_{\mathbb{R}^\times} |\omega_{os}(t_\sigma)\varphi_\sigma(\mathbf{x}_\sigma)|\, |t_\sigma|^s dt_\sigma^\times.\end{aligned}$$

Let us also define the following subsets of F^2:

$$\begin{aligned} M^+ &:= \left\{ \mathbf{x} \in F^2 \;\middle|\; Q(\mathbf{x}, \mathbf{x}) > 0 \right\}, \\ M^- &:= \left\{ \mathbf{x} \in F^2 \;\middle|\; Q(\mathbf{x}, \mathbf{x}) < 0 \right\}, \\ M^\times &:= M^- \sqcup M^+ \\ l_1 &:= \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in F^2 \;\middle|\; x \neq 0 \right\}, \\ l_2 &:= \left\{ \begin{pmatrix} 0 \\ x' \end{pmatrix} \in F^2 \;\middle|\; x' \neq 0 \right\}, \end{aligned}$$

where l_1 and l_2 are two isotropic lines, spanned by the isotropic vectors $\mathbf{e}_1 := {}^{\mathrm{t}}(1, 0)$ and $\mathbf{e}_2 := {}^{\mathrm{t}}(0, 1)$ of F^2. For $\alpha \in \mathbb{R}_{>0}$ define the K-Bessel function

$$K_s(\alpha) := \int_0^\infty e^{-\alpha(\beta+\beta^{-1})/2} \beta^s \frac{d\beta}{\beta}.$$

Lemma 4.3.3. *1. For $\mathbf{x} = x\mathbf{e}_1 \in l_1$ and $\mathrm{Re}(s) < 1$ we have*

$$J_\infty(\mathbf{x}, s) = \frac{\mathrm{N}_{F/\mathbb{Q}}(x)}{|\,\mathrm{N}_{F/\mathbb{Q}}(x)|^{1-s}} \Gamma\left(\frac{1-s}{2}\right)^N \pi^{-\frac{N(1-s)}{2}}.$$

2. For $\mathbf{x} = x'\mathbf{e}_2 \in l_2$ and $\mathrm{Re}(s) > -1$ we have

$$J_\infty(\mathbf{x}, s) = \frac{\mathrm{N}_{F/\mathbb{Q}}(x')}{|\,\mathrm{N}_{F/\mathbb{Q}}(x')|^{1+s}} \Gamma\left(\frac{1+s}{2}\right)^N \pi^{-\frac{N(1+s)}{2}}.$$

3. For $\mathbf{x} = (x, x') \in M^\times$ and any $s \in \mathbb{C}$ we have

$$\begin{aligned} J_\infty(\mathbf{x}, s) = |\mathrm{N}_{F/\mathbb{Q}}(x)|^{\frac{1+s}{2}} |\mathrm{N}_{F/\mathbb{Q}}(x')|^{\frac{1-s}{2}} \prod_\sigma \Big(&\mathrm{sgn}(x_\sigma) K_{\frac{1-s}{2}}(2\pi|x'_\sigma x_\sigma|) \\ &+ \mathrm{sgn}(x'_\sigma) K_{\frac{1+s}{2}}(2\pi|x_\sigma x'_\sigma|) \Big). \end{aligned}$$

4. For $\mathbf{x} \in M^- \subset M^\times$ we have

$$J_\infty(\mathbf{x}, 0) = 0.$$

Proof. By the definition of φ_∞ we have

$$\begin{aligned} J_\sigma(\mathbf{x}_\sigma, s) &:= \int_{-\infty}^{\infty} \exp\left(-\pi\left(\frac{x_\sigma}{t_\sigma}\right)^2 - \pi\left(x'_\sigma t_\sigma\right)^2\right)\left(\frac{x_\sigma}{t_\sigma} + x'_\sigma t_\sigma\right)\psi_\sigma(t_\sigma)|t_\sigma|^s \frac{dt_\sigma}{|t_\sigma|} \\ &= (1-\psi_\sigma(-1))\int_0^\infty \exp\left(-\pi\left(\frac{x_\sigma}{t_\sigma}\right)^2 - \pi\left(x'_\sigma t_\sigma\right)^2\right)\left(\frac{x_\sigma}{t_\sigma} + x'_\sigma t_\sigma\right)|t_\sigma|^s \frac{dt_\sigma}{t_\sigma}. \quad (4.3.5) \end{aligned}$$

Since ψ is totally odd we have $\psi_\sigma(-1) = -1$ at every archimedean place σ and this integral is nonzero. Note that $x \in F^\times$ is equivalent to $x_\sigma \neq 0$ for some σ.

1. Suppose that $x' = 0$. Then for $\mathrm{Re}(s) < 1$ we have

$$\begin{aligned} J_\sigma(\mathbf{x}_\sigma, s) &= 2\int_0^\infty \exp\left(-\pi\frac{x_\sigma^2}{t_\sigma^2}\right) x_\sigma t_\sigma^{s-1}\frac{dt_\sigma}{t_\sigma} \\ &= \frac{x_\sigma}{\pi^{\frac{1-s}{2}}|x_\sigma|^{1-s}}\Gamma\left(\frac{1-s}{2}\right). \end{aligned}$$

2. Suppose that $x = 0$. Then for $\mathrm{Re}(s) > -1$ we have

$$J_\sigma(\mathbf{x}_\sigma, s) = 2\int_0^\infty \exp\left(-\pi {x'_\sigma}^2 t^2\right) x'_\sigma t_\sigma^{1+s}\frac{dt_\sigma}{t_\sigma} = \frac{x'_\sigma}{\pi^{\frac{1+s}{2}}|x'_\sigma|^{1+s}}\Gamma\left(\frac{1+s}{2}\right).$$

3. Suppose that $x_\sigma x'_\sigma \neq 0$. Then, using the substitution $t = \sqrt{\left|\frac{x'_\sigma}{x_\sigma}\right|}u$ we have for any $s \in \mathbb{C}$

$$\begin{aligned} J_\sigma(\mathbf{x}_\sigma, s) &= 2|x_\sigma|^{\frac{1+s}{2}}|x'_\sigma|^{\frac{1-s}{2}}\int_0^\infty e^{-\pi|x'_\sigma x_\sigma|\left(u^{-2}+u^2\right)}\left(\mathrm{sgn}(x_\sigma)u^{1+s} + \mathrm{sgn}(x'_\sigma)u^{s-1}\right)\frac{du}{u} \\ &= |x_\sigma|^{\frac{1+s}{2}}|x'_\sigma|^{\frac{1+s}{2}}\left(\mathrm{sgn}(x_\sigma)K_{\frac{1+s}{2}}(2\pi|x_\sigma x'_\sigma|) + \mathrm{sgn}(x'_\sigma)K_{\frac{1-s}{2}}(2\pi|x_\sigma x'_\sigma|)\right). \end{aligned}$$

   We made use of the substitution $v = u^2$ and the fact that $K_{-s} = K_s$.

4. Since $M^- \subset M^\times$ we have by 3. that

$$J_\infty(\mathbf{x}, 0) = \prod_\sigma \left(\mathrm{sgn}(x_\sigma) + \mathrm{sgn}(x'_\sigma)\right)\sqrt{|x_\sigma x'_\sigma|}K_{\frac{1}{2}}(2\pi|x_\sigma x'_\sigma|).$$

   This can only be non-zero if $\mathrm{sgn}(x_\sigma x'_\sigma) = 1$ for all σ since $x_\sigma \neq 0$, which implies that

$$Q(\mathbf{x}, \mathbf{x}) = \mathrm{tr}_{F/\mathbb{Q}}(xx') = \sum_\sigma x_\sigma x'_\sigma = \sum_\sigma |x_\sigma x'_\sigma| \geq 0.$$

□

4.3.5 Regularization of the integral

We define the integral

$$I(\tau_1, \dots, \tau_N, \varphi, \psi, s) := \frac{(v_1 \cdots v_N)^{-\frac{1}{2}}}{\mathrm{vol}^\times(K_\infty^0)} \int_{F^\times \backslash \mathbb{A}_F^\times} \widetilde{\Theta}_{os}(g_{\underline{\tau}}, t, \varphi)\psi(t)|t|^s dt^\times$$

where $\underline{\tau} = \underline{u} + i\underline{v} = (\tau_1, \dots, \tau_N) \in \mathbb{H}^N$. In this section we want to show that the integral $I(\tau_1, \dots, \tau_N, \varphi, \psi, s)$ converges absolutely in a regularized way. Note that for the diagonal restriction to $(\tau, \dots, \tau)$ we have

$$I(\tau, \varphi, \psi, s) := I(\tau, \dots, \tau, \varphi, \psi, s) = \frac{v^{-\frac{N}{2}}}{\mathrm{vol}^\times(K_\infty^0)} \int_{F^\times \backslash \mathbb{A}_F^\times} \widetilde{\Theta}_{os}(\iota_\Delta(g_\tau), t, \varphi)\psi(t)|t|^s dt^\times.$$

Remark 4.3.1. It follows from (4.3.5) that the integral vanishes if for one place we have $\psi_\sigma(-1) = 1$ *i.e.* if ψ is not totally odd.

Since the trace form is non degenerate we can write

$$F^2 - (0,0) = l_1 \sqcup l_2 \sqcup M^\times.$$

Hence we can also split the theta series $\widetilde{\Theta}_{os}(g, t, \varphi)$ as a sum of three terms

$$\widetilde{\Theta}_{os}(g, t, \varphi) = \widetilde{\Theta}_{os}^{(l_1)}(g, t, \varphi) + \widetilde{\Theta}_{os}^{(l_2)}(g, t, \varphi) + \widetilde{\Theta}_{os}^{\times}(g, t, \varphi) \tag{4.3.6}$$

where we restrict the summation to the sets above. In view of the next proposition we call the first two terms $\widetilde{\Theta}_{os}^{(l_i)}(g, t, \varphi)$ the *singular terms* and the third one the *regular term*. The integral $I(\tau_1, \dots, \tau_N, \varphi, \psi, s)$ can also be written as a sum

$$I^{(l_1)}(\tau_1, \dots, \tau_N, \varphi, \psi, s) + I^{(l_2)}(\tau_1, \dots, \tau_N, \varphi, \psi, s) + I^\times(\tau_1, \dots, \tau_N, \varphi, \psi, s)$$

The following proposition shows that the two first integrals do not converge on the same domain. However we will see that they have a meromorphic continuation to the whole plane which allows to define the integrals for every $s \in \mathbb{C}$.

Proposition 4.3.4. *The regular term converges for every $s \in \mathbb{C}$. The singular terms converge for* $\mathrm{Re}(s) < -1$ *on l_1 and* $\mathrm{Re}(s) > 1$ *on l_2.*

Proof. Let $\nu := \mathrm{Re}(s)$ and $\mathcal{A}$ be either $M^\times$ or one of the isotropic lines l_1 or l_2. After taking the absolute value the action of $\omega(g_{\tau_k})$ is simply rescaling by $\sqrt{v_k}$ so we can assume $\tau_k = i$ for showing the convergence. Let $H^0(\infty) := (\mathbb{R}^\times)^N \times \widehat{U}(\mathfrak{f})$. By the decomposition (4.3.2) we have a surjection

$$\bigsqcup_{[\mathfrak{a}]\in \mathrm{Cl}_{\mathfrak{f}}(F)^+} t_{\mathfrak{a}} H^0(\infty) \longrightarrow F^\times \backslash \mathbb{A}_F^\times.$$

Since ψ is unitary we have

$$\begin{aligned}
&\int_{F^\times\backslash \mathbb{A}_F^\times} \sum_{\mathbf{x}\in\mathcal{A}} \Big| \left(\omega_{os}(t)\varphi\right)(\mathbf{x})\psi(t)\Big| |t|^\nu dt^\times \\
&\leq \sum_{[\mathfrak{a}]\in \mathrm{Cl}_{\mathfrak{f}}(F)^+} \int_{t_{\mathfrak{a}} H^0(\infty)} \sum_{\mathbf{x}\in\mathcal{A}} \Big| \left(\omega_{os}(t)\varphi\right)(\mathbf{x})\Big| |t|^\nu dt^\times \qquad (4.3.7)\\
&= \sum_{[\mathfrak{a}]\in \mathrm{Cl}_{\mathfrak{f}}(F)^+} |t_{\mathfrak{a}}|^\nu \int_{H^0(\infty)} \sum_{\mathbf{x}\in\mathcal{A}} \Big| \left(\omega_{os}(tt_{\mathfrak{a}})\varphi\right)(\mathbf{x})\Big| |t|^\nu dt^\times.
\end{aligned}$$

The result does not depend on the finite Schwartz function, so we can replace φ_f by $\omega_{os}(t_{\mathfrak{a}})\varphi_f$ and we have to show the convergence of

$$\int_{H^0(\infty)} \sum_{\mathbf{x}\in\mathcal{A}} \Big| \left(\omega_{os}(t)\varphi\right)(\mathbf{x})\Big| |t|^\nu dt^\times.$$

It is equivalent to that of

$$\sum_{\mathbf{x}\in\mathcal{A}} \int_{H^0(\infty)} \Big| \left(\omega_{os}(t)\varphi\right)(\mathbf{x})\Big| |t|^\nu dt^\times.$$

Since $\widehat{U}(\mathfrak{f}) \subset \widehat{\mathscr{O}}^\times$ we can bound the previous integral

$$\sum_{\mathbf{x}\in\mathcal{A}} \int_{H^0(\infty)} \Big| \left(\omega_{os}(t)\varphi\right)(\mathbf{x})\Big| |t|^\nu dt^\times \leq \sum_{\mathbf{x}\in\mathcal{A}} J_\infty(\mathbf{x},\nu)^+ J_f(\mathbf{x},\nu)^+, \qquad (4.3.8)$$

where

$$J_f(\mathbf{x},\nu)^+ = \int_{\widehat{\mathscr{O}}^\times} \Big| \left(\omega_{os}(t_f)\varphi\right)(\mathbf{x})\Big| |t_f|^\nu dt_f^\times.$$

Suppose[2] that $\mathrm{supp}(\varphi_f) \subset \widehat{\mathscr{O}}^2$, let $C_w := \sup_{\mathbf{x}\in F_w^2} |\varphi_w(\mathbf{x})|$ and $C := \prod_{w<\infty} C_w$. Note that $C_w = 1$ for almost all places w of F. Then

$$\begin{aligned}
J_f(\mathbf{x},\nu)^+ &\leq C \int_{\widehat{\mathscr{O}}^\times} \Big| \mathbf{1}_{\widehat{\mathscr{O}}}(t_f^{-1}x)\mathbf{1}_{\widehat{\mathscr{O}}}(t_f x')\Big| |t_f|^\nu dt_f^\times \\
&= C \,\mathrm{vol}^\times(\widehat{\mathscr{O}}^\times)\mathbf{1}_{\widehat{\mathscr{O}}^2}(\mathbf{x}).
\end{aligned}$$

[2]We always have $\mathrm{supp}(\varphi_f) \subset m\widehat{\mathscr{O}}^2$ for some $m \in F^\times$.

Hence we can bound (4.3.8) by

$$\sum_{\mathbf{x} \in \mathcal{A}} J_\infty(\mathbf{x}, \nu)^+ J_f(\mathbf{x}, \nu)^+ \leq C \operatorname{vol}^\times(\widehat{\mathscr{O}}^\times) \sum_{\mathbf{x} \in \mathcal{A} \cap \widehat{\mathscr{O}}_F^2} J_\infty(\mathbf{x}, \nu)^+. \tag{4.3.9}$$

1. Suppose that $\mathcal{A} = M^\times$. Following the proof of Lemma 4.3.3 we see that

$$J_\infty(\mathbf{x}, \nu)^+ \leq |\mathrm{N}(x)|^{\frac{1+\nu}{2}} |\mathrm{N}(x')|^{\frac{1-\nu}{2}} \prod_\sigma \left(K_{\frac{1-\nu}{2}}(2\pi |x'_\sigma x_\sigma|) + K_{\frac{1+\nu}{2}}(2\pi |x_\sigma x'_\sigma|) \right).$$

   For $\nu \in \mathbb{R}$ we have the bound

$$K_\nu(\alpha) < 2^{2(|\nu|+1)} \left(1 + \frac{\Gamma(|\nu|+1)}{\alpha^{|\nu|+1}} \right) e^{-\alpha},$$

   see [O'S18, Lemma. 1.3] for example. Thus we can bound

$$J_\infty(\mathbf{x}, \nu)^+ \leq G(\mathbf{x}) e^{-\sum x_\sigma x'_\sigma} \tag{4.3.10}$$

   where $G(\mathbf{x})$ is a rational function in $\mathbf{x}$. Since we sum over $M^\times$ none of the x_σ or x'_σ are zero. Hence the right hand side of (4.3.10) is now rapidly decreasing on $M^\times$ and the sum (4.3.9) converges.

2. Now suppose that $\mathcal{A} = l_2$. Following the proof of Lemma 4.3.3 we can bound (4.3.9) by

$$\sum_{\mathbf{x} \in l_2} J_\infty(\mathbf{x}, \nu)^+ \leq \Gamma\left(\frac{1+\nu}{2}\right)^N \pi^{-N\frac{1+\nu}{2}} \sum_{x \in \mathscr{O}^*} \frac{1}{|\mathrm{N}(x)|^\nu},$$

   which converges for $\nu > 1$.

3. The case $\mathcal{A} = l_1$ is similar.

□

We recall that Λ was defined in (4.1.5) by $\Lambda(s) = \Gamma\left(\frac{1+s}{2}\right)^N \pi^{-\frac{N(1+s)}{2}}$.

Proposition 4.3.5. *For the singular terms we have*

$$I^{(l_1)}(\tau_1, \ldots, \tau_N, \varphi, \psi, s) = 2^{1-N} \Lambda(-s)(v_1 \cdots v_N)^{\frac{s}{2}} \zeta_f(\varphi_1, \psi^{-1}, -s)$$
$$I^{(l_2)}(\tau_1, \ldots, \tau_N, \varphi, \psi, s) = 2^{1-N} \Lambda(s)(v_1 \cdots v_N)^{-\frac{s}{2}} \zeta_f(\varphi_2, \psi, s),$$

where φ_1 and φ_2 in $\mathscr{S}(\mathbb{A}_F)$ are defined by $\varphi_1(x) = \varphi\begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\varphi_2(x') = \varphi\begin{pmatrix} 0 \\ x' \end{pmatrix}$. In particular they both have a continuation to the whole plane.

Proof. For the singular term $I^{(l_1)}(\tau_1,\dots,\tau_N,\varphi,\psi,s)$ we know that we have absolute convergence for every $\mathrm{Re}(s)<-1$. Hence we can exchange summation and integration and unfold the integral to get

$$\begin{aligned}&\int_{F^\times\backslash\mathbb{A}_F^\times}\sum_{x\in F^\times}\Big(\omega_{os}(g_{\underline{\tau}},t)\varphi\Big)(x\mathbf{e}_1)\psi(t)|t|^s dt^\times\\ &=(v_1\cdots v_N)^{\frac{1}{2}}\int_{F^\times\backslash\mathbb{A}_F^\times}\sum_{x\in F^\times}\varphi_1(\sqrt{\underline{v}}t^{-1}x)\psi(t)|t|^s dt^\times\\ &=(v_1\cdots v_N)^{\frac{1}{2}}\int_{\mathbb{A}_F^\times}\varphi_1(\sqrt{\underline{v}}t^{-1}x)\psi(t)|t|^s dt^\times.\end{aligned}$$

After the substitution $(t_\infty,t_f)\mapsto z=(\sqrt{\underline{v}}t_\infty^{-1},t_f^{-1})$ we get

$$\begin{aligned}&=(v_1\cdots v_N)^{\frac{1}{2}+\frac{s}{2}}\int_{\mathbb{A}_F^\times}\varphi_1(z)\psi(z)^{-1}|z|^{-s}dz^\times\\ &=(v_1\cdots v_N)^{\frac{1}{2}+\frac{s}{2}}\zeta(\varphi_1,\psi^{-1},-s).\end{aligned}$$

The result follows after multiplying by $\mathrm{vol}^\times(K_\infty^0)^{-1}(v_1\cdots v_N)^{-\frac{1}{2}}$, where $\mathrm{vol}^\times(K_\infty^0)=2^{N-1}$, and using the computation (4.1.5) of the archimedean zeta function. For the second term the computation is similar.

Since we supposed that the character is unitary and odd, its restriction to the adèles of norm 1 is never trivial. Hence by [Bum97, Proposition. 3.16], the continued function is entire. □

Proposition 4.3.6. *Suppose that φ_1 or φ_2 vanishes. The integral $I(\tau,\varphi,\psi,s)$ can be analytically continued to the whole plane. The value $I(\tau,\varphi,\psi)\coloneqq I(\tau,\varphi,\psi,s)\big|_{s=0}$ at $s=0$ is*

$$I(\tau,\varphi,\psi)=2^{1-N}\zeta_f(\varphi_1,\psi^{-1},0)+2^{1-N}\zeta_f(\varphi_2,\psi,0)+\sum_{n\in\mathbb{Q}_{>0}}\left(\int_{C\otimes\psi}\Theta_n(v,\varphi_f)\right)e^{2i\pi n\tau},$$

where κ is 2 if $-1\in K_f\cap H(\mathbb{Q})^+$ and 1 otherwise.

Proof. By Proposition 4.3.4 the singular terms converge on two disjoint half planes. Hence the assumption that φ_1 or φ_2 vanishes guarantees that one of the singular terms is zero.

As in (4.3.6) let us write the theta series as the sum

$$I(\tau,\varphi,\psi,s)=I^-(\tau,\varphi,\psi,s)+I^{(l_1)}(\tau,\varphi,\psi,s)+I^{(l_2)}(\tau,\varphi,\psi,s)+I^+(\tau,\varphi,\psi,s)$$

where the + (respectively the −) means that we sum over positive (respectively negative) vectors. In particular $I^\times=I^-+I^+$.

For the term $I^-(\tau,\varphi,\psi,s)$ we know that we have absolute convergence for every $s\in\mathbb{C}$. Hence we can exchange summation and integration at $s = 0$ and we get the bound (as in (4.3.7))

$$\begin{aligned} I^{-}(\tau,\varphi,\psi) &= v^{-\frac{N}{2}} 2^{1-N} \int_{F^\times \backslash \mathbb{A}_F^\times} \widetilde{\Theta}_{os}^{-}(\iota_\Delta(g_\tau), t, \varphi)\psi(t) dt^\times \\ &\leq v^{-\frac{N}{2}} 2^{1-N} \sum_{\mathfrak{a} \in \mathrm{Cl}_{\mathfrak{f}}(F)^+} \sum_{\mathbf{x} \in M^-} \int_{t_{\mathfrak{a}} H^0(\infty)} \Big(\omega_{os}(\iota_\Delta(g_\tau), t)\varphi\Big)(\mathbf{x})\psi(t) dt^\times \\ &= 2^{1-N} \sum_{\mathfrak{a} \in \mathrm{Cl}_{\mathfrak{f}}(F)^+} \sum_{\mathbf{x} \in M^-} e^{i\pi u Q(\mathbf{x},\mathbf{x})} J_\infty(\sqrt{v}\mathbf{x}, 0) \int_{t_{\mathfrak{a}} \widehat{U}(\mathfrak{f})} \Big(\omega_{os}(t)\varphi_f\Big)(\sqrt{v}\mathbf{x})\psi_f(t) dt^\times. \end{aligned} \tag{4.3.11}$$

The vanishing of the integral (4.3.11) follows from Lemma 4.3.3.

For the term $I^{+}(\tau,\varphi,\psi,s)$ we also have absolute convergence for every $s \in \mathbb{C}$. Thus, using Proposition 4.3.2 we can write

$$\begin{aligned} I^{+}(\tau,\varphi,\psi) &= v^{-\frac{N}{2}} 2^{1-N} \int_{F^\times \backslash \mathbb{A}_F^\times} \widetilde{\Theta}_{os}^{+}(\iota_\Delta(g_\tau), t, \varphi)\psi(t) dt^\times \\ &= \sum_{[\mathfrak{a}] \in \mathrm{Cl}_{\mathfrak{f}}(F)^+} \psi(\mathfrak{a}) v^{-\frac{N}{2}} \int_{\Gamma_{\mathfrak{a}} \backslash \mathbb{D}_0^+} \widetilde{\Theta}_{os}^{+}(\iota_\Delta(g_\tau), (t_\infty, t_{\mathfrak{a}}), \varphi) dt_\infty^\times \\ &= \int_{C \otimes \psi} \Theta_{KM}^{+}(\tau, \varphi_f), \end{aligned}$$

where $\Theta_{KM}^{+}(\tau,\varphi_f)$ is the restriction to $\Theta_{KM}(\tau,\varphi_f)$ to positive vectors. By Lemma 2.2.1 we then have

$$\int_{C \otimes \psi} \Theta_{KM}^{+}(\tau,\varphi_f) = \sum_{n \in \mathbb{Q}_{>0}} \left(\int_{C \otimes \psi} \Theta_n(v, \varphi_f) \right) e^{2i\pi n \tau}.$$

Finally, the constant term of $I(\tau,\varphi,\psi)$ is

$$I^{(l_1)}(\tau,\varphi,\psi,0) + I^{(l_2)}(\tau,\varphi,\psi,0)$$

and was computed in Proposition 4.3.5. In particular it was shown that $I^{(l_1)}(\tau,\varphi,\psi,s)$ and $I^{(l_2)}(\tau,\varphi,\psi,s)$ have an analytic continuation to the whole plane. □

4.3.6 Orientations

Before showing that the Fourier coefficients are intersection numbers, we need to fix some orientations. We fixed the orientations

$$o(F_{\mathbb{R}}^2) := \mathbf{e}_{\sigma_1} \wedge \cdots \wedge \mathbf{e}_{\sigma_N} \wedge \mathbf{f}_{\sigma_1} \wedge \cdots \wedge \mathbf{f}_{\sigma_N}, \qquad o(z_0) := \mathbf{f}_{\sigma_1} \wedge \cdots \wedge \mathbf{f}_{\sigma_N}$$

of $F_{\mathbb{R}}^2$ and of z_0, and we can use it to orient $\mathbb{D}^+$ and $\mathbb{D}^+_{\mathbf{x}}$ as in Subsection 2.2.3. Let us also orient $\mathbb{D}^+_0$ at z_0. We restrict the map h to a small interval around 1 such that its image lies in $\mathcal{V}_{z_0}$, the chart around z_0 consisting of planes that are transverse to z_0. More precisely let $\epsilon > 1$ such that

$$h\colon \left(\frac{1}{\epsilon}, \epsilon\right)^N \longrightarrow \mathbb{D}^+_0 \cap \mathcal{V}_{z_0}.$$

The differential at $(1, \dots, 1)$ gives a map

$$dh\colon \mathbb{R}^N_{>0} \longrightarrow T_{z_0}\mathbb{D}^+_0 \subset z_0^\vee \otimes z_0^\perp,$$

and we use the standard orientation $o(\mathbb{R}^N) = \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_N}$ to orient the tangent space. Hence orienting $\mathbb{D}^+_0$ amounts to a choice of an ordering of the places σ_k.

Lemma 4.3.7. *We have* $dh\left(\frac{\partial}{\partial t_i}\right) = \mathbf{f}_i^\vee \otimes \mathbf{e}_i$.

Proof. Without loss of generality we take $i = 1$. Let $\rho\colon \mathbb{R} \longrightarrow \left(\frac{1}{\epsilon}, \epsilon\right)^N$ be the curve $\rho(u) = (e^u, 1 \cdots, 1)$ be the curve representing $\frac{\partial}{\partial t_1}$. We have

$$\begin{aligned} h(\rho(u))z_0 &= h(\gamma(u))\,\mathrm{span}\big\langle \mathbf{f}_1, \dots, \mathbf{f}_N \big\rangle \\ &= \mathrm{span}\Big\langle \frac{e^u - e^{-u}}{2}\mathbf{e}_1 + \frac{e^u + e^{-u}}{2}\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_N \Big\rangle \\ &= \mathrm{span}\Big\langle \frac{e^u - e^{-u}}{e^u + e^{-u}}\mathbf{e}_1 + \mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_N \Big\rangle =: z(u). \end{aligned}$$

As an element of $z_0^\vee \otimes z_0^\perp$ we have

$$z(u) = \frac{e^u - e^{-u}}{e^u + e^{-u}}\mathbf{f}_1^\vee \otimes \mathbf{e}_1.$$

Then $dh\left(\frac{\partial}{\partial t}\right)$ is equal to

$$\frac{d}{du}z(u)\Big|_{u=0} = \mathbf{f}_1^\vee \otimes \mathbf{e}_1.$$

□

4.3.7 The positive Fourier coefficients as intersection numbers

In this subsection we show that the Fourier coefficients $\int_{C\otimes\psi} \Theta_n(v, \varphi_f)$ are intersection numbers, it is the content of Proposition 4.3.10.

Proposition 4.3.8. *Let $\mathbf{x} \in F^2$ with $Q(\mathbf{x}, \mathbf{x}) > 0$. Then*

$$|\mathbb{D}_0^+ \cap \mathbb{D}_{\mathbf{x}}^+| = \begin{cases} 1 & \text{if } \operatorname{sgn}(x_\sigma x'_\sigma) = 1 \text{ for all } \sigma \\ 0 & \text{otherwise,} \end{cases}$$

and it only depends on the $\mathscr{O}^{\times,+}$ orbit of $\mathbf{x}$, where we view $\mathscr{O}^{\times,+}$ in Γ'_{h_i} embedded via $h\colon F_\infty^\times \hookrightarrow H(\mathbb{R})^+$. Furthermore the intersection is transversal and we have

$$\langle \mathbb{D}_{\mathbf{x}}^+, \mathbb{D}_0^+ \rangle = \begin{cases} (-1)^N \operatorname{sgn} \mathrm{N}(x) & \text{if } \operatorname{sgn}(x_\sigma x'_\sigma) = 1 \text{ for all } \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The set $\mathbb{D}_0^+ \cap \mathbb{D}_{\mathbf{x}}^+$ is precisely the zero locus of $\oplus_\sigma s_{\mathbf{x}_\sigma} \colon \mathbb{D}_0^+ \longrightarrow \mathbb{D}_0^+ \times F_\infty$, where

$$s_{\mathbf{x}_\sigma}(t_\sigma) = \left(t_\sigma, \frac{t_\sigma^{-1} x_\sigma - t_\sigma x'_\sigma}{\sqrt{2}} \right) \in \mathbb{R}_{>0} \times F_\sigma,$$

see the diagram (4.3.4). This section vanishes exactly when $t_\sigma^2 = \frac{x_\sigma}{x'_\sigma}$ for all σ. This has precisely one solution when $\operatorname{sgn}(x_\sigma x'_\sigma) = 1$ for all σ.

Suppose that the intersection point is z_0. We have $o(N_{z_0}\mathbb{D}_{\mathbf{x}}^+) = (\mathbf{f}_1^\vee \otimes \mathbf{x}) \wedge \cdots \wedge (\mathbf{f}_N^\vee \otimes \mathbf{x})$ and $o(T_{z_0}\mathbb{D}_0^+) = (\mathbf{f}_1^\vee \otimes \mathbf{e}_1) \wedge \cdots \wedge (\mathbf{f}_N^\vee \otimes \mathbf{e}_N)$. Since $z_0 \in \mathbb{D}_{\mathbf{x}}^+$ we have $\mathbf{x} \in z_0^\perp$. Hence we can write $\mathbf{x} = {}^{\mathrm{t}}(x, x) = \sum_j x_j \mathbf{e}_j$ and

$$\begin{aligned} q_{z_0}(o(T_{z_0}\mathbb{D}_0^+), o(N_{z_0}\mathbb{D}_{\mathbf{x}}^+)) &= \det\left(-Q(\mathbf{f}_i^\vee, \mathbf{f}_j^\vee) Q(\mathbf{e}_i, \mathbf{x})\right)_{ij} \\ &= (-1)^N 2^N \prod_j x_j = (-1)^N 2^N \mathrm{N}(x), \end{aligned}$$

where the factor 2^N comes from $Q(\mathbf{f}_i^\vee, \mathbf{f}_j^\vee) = 2\delta_{ij}$. Hence the intersection is transversal and

$$\langle \mathbb{D}_{\mathbf{x}}^+, \mathbb{D}_0^+ \rangle = (-1)^N \operatorname{sgn} \mathrm{N}(x).$$

Now suppose that the intersection point is $z \neq z_0$. Then for some $t_\infty = (t_1, \ldots, t_N)$ with $t_j > 0$ we have $z = h(t_\infty) z_0$ and thus

$$z_0 = h(t_\infty)^{-1} \left(\mathbb{D}_0^+ \cap \mathbb{D}_{\mathbf{x}}^+ \right) = \mathbb{D}_0^+ \cap \mathbb{D}_{h(t_\infty)^{-1}\mathbf{x}}^+.$$

The intersection at z_0 is transversal and since $h(t_\infty)$ is orientation preserving (since $t_j > 0$) the sign of $\langle \mathbb{D}_{\mathbf{x}}^+, \mathbb{D}_0^+ \rangle$ equals the sign of

$$\langle \mathbb{D}_{h(t_\infty)^{-1}\mathbf{x}}^+, \mathbb{D}_0^+ \rangle = (-1)^N \operatorname{sgn} \prod t_j^{-1} x_j = (-1)^N \operatorname{sgn} \mathrm{N}(x).$$

□

Consider the set

$$\mathcal{L}_{h_i}(\varphi_f) = \left\{ \mathbf{x} \in F_{\mathbb{Q}}^2 \middle| \, \varphi_f(h_i^{-1}\mathbf{x}) \neq 0 \right\}.$$

There exists an $m \in F^\times$ such that $\mathrm{supp}(\varphi_f) \subset m\widehat{\mathscr{O}}^2$, hence there is an $m_i \in F^\times$ so that $\mathcal{L}_{h_i}(\varphi_f) \subset m_i\mathscr{O}^2$. Let Γ'_{h_i} be the subgroup defined in 2.2.8.

Proposition 4.3.9. *Let $\mathbf{x} \in \mathcal{L}_{h_i}(\varphi_f)$. The intersection $\mathbb{D}_{\mathfrak{a}}^+ \cap \mathbb{D}_{\mathbf{y}}^+$ is non-zero for only finitely many orbits $\mathscr{O}^{\times,+}\mathbf{y} \in \mathscr{O}^{\times,+}\backslash\Gamma'_{h_i}\mathbf{x}$.*

Proof. Let $Q(\mathbf{x},\mathbf{x}) = 2n$ for some $n \in \mathbb{Q}$. For every $\mathbf{y} \in \Gamma'_{h_i}\mathbf{x}$ we have $Q(\mathbf{y},\mathbf{y}) = 2n$. Since $\mathbb{D}_{\mathfrak{a}}^+ \cap \mathbb{D}_{\mathbf{y}}^+ = h_{\mathfrak{a}}\left(\mathbb{D}_0^+ \cap \mathbb{D}_{h_{\mathfrak{a}}^{-1}\mathbf{y}}^+\right)$ it is enough to prove the statement for the intersection $\mathbb{D}_0^+ \cap \mathbb{D}_{\mathbf{y}}^+$. We begin by showing that if $\mathcal{K} \subset \mathbb{D}^+$ is a compact subset then $\mathcal{K} \cap \mathbb{D}_{\mathbf{y}}^+$ is non-empty for only finitely many $\mathbf{y} \in \Gamma'_{h_i}\mathbf{x}$. For a negative plane $z \in \mathbb{D}^+$ and $\mathbf{y} \in F^2$ we have

$$Q(\mathbf{y},\mathbf{y}) = Q(\mathbf{y}_{z^\perp},\mathbf{y}_{z^\perp}) + Q(\mathbf{y}_z,\mathbf{y}_z)$$

where $\mathbf{y} = \mathbf{y}_z + \mathbf{y}_{z^\perp}$ be the orthogonal decomposition of $\mathbf{y}$ with respect to the splitting $z \oplus z^\perp$. The Siegel majorant

$$Q_z^+(\mathbf{y},\mathbf{y}) \coloneqq Q(\mathbf{y}_{z^\perp},\mathbf{y}_{z^\perp}) - Q(\mathbf{y}_z,\mathbf{y}_z)$$

is a positive definite quadratic form. If $z \in \mathbb{D}_{\mathbf{y}}^+$ then $z \subset \mathbf{y}^\perp$ and $Q_z^+(\mathbf{y},\mathbf{y}) = Q(\mathbf{y},\mathbf{y}) = 2n$. Since the Siegel majorant is positive definite we can find real numbers $M_z > m_z > 0$ such that $m_z\|\mathbf{y}\|^2 \leq Q_z^+(\mathbf{y},\mathbf{y}) \leq M_z\|\mathbf{y}\|^2$ where $\|-\|$ is the Euclidean norm on $\mathbb{R}^{2N}$. Since $\mathcal{K}$ is compact and Q_z^+ is continuous in z we can also find constants $M_{\mathcal{K}} > m_{\mathcal{K}} > 0$ such that

$$m_{\mathcal{K}}\|\mathbf{y}\|^2 \leq Q_z^+(\mathbf{y},\mathbf{y}) \leq M_{\mathcal{K}}\|\mathbf{y}\|^2$$

for every $z \in \mathcal{K}$. Hence for $z \in \mathbb{D}_{\mathbf{y}}^+ \cap \mathcal{K}$ we have

$$\frac{2n}{M_{\mathcal{K}}} = \frac{Q_z^+(\mathbf{y},\mathbf{y})}{M_{\mathcal{K}}} \leq \|\mathbf{y}\|^2 \leq \frac{Q_z^+(\mathbf{y},\mathbf{y})}{m_{\mathcal{K}}} = \frac{2n}{m_{\mathcal{K}}}.$$

There are only finitely many vectors $\mathbf{y}$ of bounded norm in the lattice $\mathcal{L}_{h_i}(\varphi_f)$, let alone in an orbit $\Gamma'_{h_i}\mathbf{x}$.

Now let $\mathbb{D}_0^{+,1} \subset \mathbb{D}_0^{+}$ be the subset of elements of norm 1, so that we have a diffeomorphism

$$\begin{aligned}\mathbb{D}_0^{+} &\longrightarrow \mathbb{D}_0^{+,1} \times \mathbb{R}_{>0} \\ (t_{\sigma_1}, \ldots, t_{\sigma_N}) &\longmapsto \left[\left(t_{\sigma_1}, \ldots, t_{\sigma_{N-1}}, \frac{1}{t_{\sigma_1} \cdots t_{\sigma_{N-1}}}\right), t_{\sigma_1} \cdots t_{\sigma_N}\right]\end{aligned}$$

For $T > 1$ we define

$$\begin{aligned}\mathbb{D}_0^{+}(T) &\coloneqq \mathbb{D}_0^{+,1} \times \left[\frac{1}{T}, T\right] \\ &\simeq \left\{ (t_1, \ldots, t_N) \in \mathbb{R}_{>0}^N \middle| \, \frac{1}{T} \leq t_1 \cdots t_N \leq T \right\}.\end{aligned}$$

The group $\mathscr{O}^{\times,+}$ preserves $\mathbb{D}_0^{+,1}$, and by Dirichlet's unit Theorem the quotient is compact. Hence we can find $\mathscr{F} \subset \mathbb{D}_0^{+,1}$ such that $\overline{\mathscr{F}}$ is compact and

$$\mathbb{D}_0^{+,1} = \bigsqcup_{\lambda \in \mathscr{O}^{\times,+}} \lambda \mathscr{F}.$$

We also set $\mathscr{F}_T \coloneqq \mathscr{F} \times \left[\frac{1}{T}, T\right]$ so that

$$\mathbb{D}_0^{+}(T) = \bigsqcup_{\lambda \in \mathscr{O}^{\times,+}} \lambda \mathscr{F}_T.$$

Suppose that there is $T > 1$ that only depends on n such that

$$\mathbb{D}_0^{+} \cap \mathbb{D}_{\mathbf{y}}^{+} = \mathbb{D}_0^{+}(T) \cap \mathbb{D}_{\mathbf{y}}^{+}. \tag{4.3.12}$$

Since $\overline{\mathscr{F}_T}$ is compact there are finitely many distinct vectors $\mathbf{y}_1, \ldots, \mathbf{y}_k$ such that $|\mathscr{F}_T \cap \mathbb{D}_{\mathbf{y}_i}^{+}|$ is non-zero. For $\mathbf{y} \in F^2$ we then have

$$\begin{aligned}\mathbb{D}_0^{+} \cap \mathbb{D}_{\mathbf{y}}^{+} &= \mathbb{D}_0^{+}(T) \cap \mathbb{D}_{\mathbf{y}}^{+} \\ &= \bigsqcup_{\lambda \in \mathscr{O}^{\times,+}} |\lambda \mathscr{F}_T \cap \mathbb{D}_{\mathbf{y}}^{+}| \\ &= \bigsqcup_{\lambda \in \mathscr{O}^{\times,+}} \lambda \left(\mathscr{F}_T \cap \mathbb{D}_{\lambda^{-1}\mathbf{y}}^{+} \right).\end{aligned}$$

This intersection is empty if $\mathbf{y}$ lies in none of the orbits $\mathscr{O}^{\times,+}\mathbf{y}_i$.

Finally let us prove (4.3.12). Let $\mathbf{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$ be a vector in $\mathcal{L}_{h_i}$ with $Q(\mathbf{y}, \mathbf{y}) = 2\,\mathrm{tr}_{F/\mathbb{Q}}(yy') = 2n$.

If $t_\infty \in \mathbb{D}_0^+ \cap \mathbb{D}_{\mathbf{y}}^+$, then $y_\sigma y'_\sigma > 0$ and

$$\prod_\sigma t_\sigma = \prod_\sigma \sqrt{\frac{y_\sigma}{y'_\sigma}} = \sqrt{\left|\frac{\mathrm{N}(y)}{\mathrm{N}(y')}\right|}.$$

Since $\mathcal{L}_{h_i} \subset m_i \mathscr{O}^2$ for some $m_i \in F^\times$ we have $|\,\mathrm{N}(y)| \geq |\,\mathrm{N}(m_i)| > 0$. On the other hand, since $y_\sigma y'_\sigma > 0$ we can use the inequality between arithmetic and geometric mean to show that

$$\frac{2n}{N} = \frac{Q(\mathbf{y},\mathbf{y})}{N} \geq 2|\,\mathrm{N}(y)\,\mathrm{N}(y')|^{\frac{1}{N}} \geq 2|\,\mathrm{N}(y)\,\mathrm{N}(m_i)|^{\frac{1}{N}}.$$

Hence

$$\frac{1}{|\,\mathrm{N}(m_i)|}\left(\frac{n}{N}\right)^N \geq |\,\mathrm{N}(y)| \geq |\,\mathrm{N}(m_i)|.$$

By replacing y by y' we obtain the same bound for $\mathrm{N}(y')$, which shows that we can take

$$T = \frac{1}{|\,\mathrm{N}(m_i)|}\left(\frac{n}{N}\right)^{\frac{N}{2}}.$$

□

Every connected cycle $C_{\mathfrak{a}}$ land in exactly one connected component M_{h_i} of M_K. We can then define the intersection numbers

$$\left\langle C_{\mathbf{x}}(h_i), C_{\mathfrak{a}}\right\rangle := \frac{1}{\kappa} \sum_{[\mathbf{y}] \in \mathscr{O}^{\times,+} \backslash \Gamma'_{h_i}\mathbf{x}} \left\langle \mathbb{D}_{\mathbf{y}}^+, \mathbb{D}_{\mathfrak{a}}^+\right\rangle \tag{4.3.13}$$

which are well-defined by the previous proposition. We also define

$$\left\langle C_n(\varphi_f), C \otimes \psi\right\rangle := \sum_{[\mathfrak{a}] \in \mathrm{Cl}_{\mathfrak{f}}(F)^+} \sum_{i=1}^{r} \sum_{\substack{\mathbf{x} \in \Gamma'_{h_i} \backslash F_{\mathbb{Q}}^2 \\ Q(\mathbf{x},\mathbf{x}) = 2n}} \psi(\mathfrak{a})\varphi_f(h_i^{-1}\mathbf{x})\left\langle C_{\mathbf{x}}(h_i), C_{\mathfrak{a}}\right\rangle, \tag{4.3.14}$$

where for every class $[\mathfrak{a}]$ the intersection number $\left\langle C_{\mathbf{x}}(h_i), C_{\mathfrak{a}}\right\rangle$ is non-zero for at most one h_i, corresponding to the connected component of M_K that contains $C_{\mathfrak{a}}$.

Remark 4.3.2. Suppose that $-1 \in \Gamma'_{h_i}$, thus $\kappa = 2$. Since $-1 \notin \mathscr{O}^{\times,+}$, every orbit appears twice, we sum over $\mathbf{y} = \mathscr{O}^{\times,+}\mathbf{x}$ and $\mathbf{y} = -\mathscr{O}^{\times,+}\mathbf{x}$ in (4.3.13). Since $\mathbb{D}_{-\mathbf{y}}^+ = \mathbb{D}_{\mathbf{y}}^+$ we are counting the intersection number $\left(\mathbb{D}_{\mathbf{y}}^+, \mathbb{D}_{\mathfrak{a}}^+\right)$ twice and this is why we have to divide by $\kappa = 2$.

Proposition 4.3.10. *We have*

$$\int_{C\otimes\psi} \Theta_n(v,\varphi_f) = (-1)^N \kappa \big\langle C_n(\varphi_f), C\otimes\psi \big\rangle.$$

Proof. It is enough to show that

$$\int_{C_{\mathfrak{a}}} \Theta_n(v,\varphi_f) = (-1)^N \kappa \big\langle C_n(\varphi_f), C_{\mathfrak{a}} \big\rangle.$$

For $\mathbf{x} \in F^2$ with $Q(\mathbf{x},\mathbf{x}) > 0$ we first show that

$$\int_{\mathbb{D}_0^+} \varphi^0(\mathbf{x}) = (-1)^N \big\langle \mathbb{D}_{\mathbf{x}}^+, \mathbb{D}_0^+ \big\rangle. \tag{4.3.15}$$

We have

$$\begin{aligned}\int_{\mathbb{D}_0^+} \varphi^0(\mathbf{x}) &= 2^{-N} e^{\pi Q(\mathbf{x},\mathbf{x})} J_\infty(\mathbf{x},0) \\ &= 2^{-N}\sqrt{\mathrm{N}(xx')} e^{\pi Q(\mathbf{x},\mathbf{x})} \prod_\sigma \big(\operatorname{sgn}(x_\sigma) + \operatorname{sgn}(x'_\sigma)\big) K_{\frac{1}{2}}(2\pi|x_\sigma x'_\sigma|).\end{aligned} \tag{4.3.16}$$

It follows from Lemma 4.3.9 that the intersection $\mathbb{D}_0^+ \cap \mathbb{D}_{\mathbf{x}}^+$ is empty if and only if $\operatorname{sgn}(x_\sigma) \neq \operatorname{sgn}(x_\sigma)$ for some σ. Since x_σ and x'_σ are non-zero (otherwise x would be 0) it follows that this is equivalent to the vanishing of (4.3.16). Thus we can suppose that $\operatorname{sgn}(x_\sigma) = \operatorname{sgn}(x_\sigma)$ for all σ. Using the equality $K_{\frac{1}{2}}(\alpha) = \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha}$ we find that

$$\int_{\mathbb{D}_0^+} \varphi^0(\mathbf{x}) = \operatorname{sgn} \mathrm{N}(x) = (-1)^N \big\langle \mathbb{D}_{\mathbf{x}}, \mathbb{D}_0^+ \big\rangle.$$

By the invariance property of the Kudla-Millson form we have

$$\begin{aligned}\int_{\mathbb{D}_{\mathfrak{a}}^+} \varphi^0(\mathbf{x}) &= \int_{h_{\mathfrak{a}}\mathbb{D}_0^+} \varphi^0(\mathbf{x}) \\ &= \int_{\mathbb{D}_0^+} \varphi^0(h_{\mathfrak{a}}^{-1}\mathbf{x}) \\ &= (-1)^N \big\langle \mathbb{D}_{h_{\mathfrak{a}}^{-1}\mathbf{x}}^+, \mathbb{D}_0^+ \big\rangle \\ &= (-1)^N \big\langle \mathbb{D}_{\mathbf{x}}^+, \mathbb{D}_{\mathfrak{a}}^+ \big\rangle.\end{aligned} \tag{4.3.17}$$

Hence (4.3.15) also holds for $\mathbb{D}_{\mathfrak{a}}^+$ instead of $\mathbb{D}_0^+$. Then we have

$$\begin{aligned}\int_{C_{\mathfrak{a}}} \Theta_n(v,\varphi_f) &= \sum_{i=1}^{r} \sum_{\substack{\mathbf{x}\in\Gamma'_{h_i}\backslash F_{\mathbb{Q}}^2 \\ Q(\mathbf{x},\mathbf{x})=2n}} \varphi_f(h_i^{-1}\mathbf{x}) \int_{C_{\mathfrak{a}}} \sum_{\mathbf{y}\in\Gamma'_{h_i}\mathbf{x}} \varphi^0(\sqrt{v}\mathbf{y}) \\ &= \sum_{i=1}^{r} \sum_{\substack{\mathbf{x}\in\Gamma'_{h_i}\backslash F_{\mathbb{Q}}^2 \\ Q(\mathbf{x},\mathbf{x})=2n}} \varphi_f(h_i^{-1}\mathbf{x}) \sum_{\mathbf{y}\in\mathscr{O}^{\times,+}\backslash\Gamma'_{h_i}\mathbf{x}} \int_{\mathbb{D}_{\mathfrak{a}}^+} \varphi^0(\sqrt{v}\mathbf{y}).\end{aligned}$$

Since $\mathbb{D}^+_{\sqrt{v}\mathbf{y}} = \mathbb{D}^+_{\mathbf{y}}$ we then get

$$\begin{aligned}\int_{C_{\mathfrak{a}}} \Theta_n(v,\varphi_f) &= (-1)^N \sum_{i=1}^{r} \sum_{\substack{\mathbf{x}\in\Gamma'_{h_i}\backslash F_{\mathbb{Q}}^2 \\ Q(\mathbf{x},\mathbf{x})=2n}} \varphi_f(h_i^{-1}\mathbf{x}) \sum_{\mathbf{y}\in\mathscr{O}^{\times,+}\backslash\Gamma'_{h_i}\mathbf{x}} \left\langle \mathbb{D}_{\mathbf{y}}^+, \mathbb{D}_{\mathfrak{a}}^+ \right\rangle \\ &= (-1)^N \kappa \sum_{i=1}^{r} \sum_{\substack{\mathbf{x}\in\Gamma'_{h_i}\backslash F_{\mathbb{Q}}^2 \\ Q(\mathbf{x},\mathbf{x})=2n}} \varphi_f(h_i^{-1}\mathbf{x}) \left\langle C_{\mathbf{x}}(h_i), C_{\mathfrak{a}} \right\rangle.\end{aligned}$$

□

Hence with Proposition 4.3.6 we get

$$I(\tau,\varphi,\psi) = 2^{1-N}\zeta_f(\varphi_1,\psi^{-1},0) + 2^{1-N}\zeta_f(\varphi_2,\psi,0) + (-1)^N\kappa \sum_{n\in\mathbb{Q}_{>0}} \left\langle C_n(\varphi_f), C\otimes\psi \right\rangle e^{2i\pi n\tau}.$$

4.3.8 Change of model

Recall that $W_F^0 = X_F^0 \oplus X_F^0 \simeq F^4$ is a 4-dimensional symplectic space over F, whose restriction of scalars was $W_{\mathbb{Q}} = X_{\mathbb{Q}} \oplus X_{\mathbb{Q}}$. The sympectic form on W_F^0 is given by the skew symmetric-matrix $\begin{pmatrix} 0 & A(Q^0) \\ -A(Q^0) & 0 \end{pmatrix} \in \mathrm{Mat}_4(F)$ where $A(Q^0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence the symplectic group of W_F^0 is

$$\mathrm{Sp}(W_F^0) = \left\{ g \in \mathrm{GL}_4(F) \,\middle|\, {}^{\mathrm{t}}g \begin{pmatrix} & A(Q^0) \\ -A(Q^0) & \end{pmatrix} g = \begin{pmatrix} & A(Q^0) \\ -A(Q^0) & \end{pmatrix} \right\}.$$

Since $F^\times \subset \mathrm{SO}(F^2)$ we have two different models for the pair $\mathrm{SL}_2(F) \times F^\times$ in $\mathrm{Sp}(W_F^0)$: on the one hand we can use the linear one for $\mathrm{GL}_2(F) \times F^\times$, on the other hand we can use the orthosymplectic model for $\mathrm{SL}_2(F) \times \mathrm{SO}(F^2)$; see Subsection 2.1.9. These models correspond to two different embedding $\iota_{os}, \iota_l \colon \mathrm{SL}_2(F) \times F^\times \hookrightarrow \mathrm{Sp}(W_F^0)$ given by

$$\iota_{os}(g,t) = \begin{pmatrix} at & & bt & \\ & at^{-1} & & bt^{-1} \\ ct & & dt & \\ & ct^{-1} & & dt^{-1} \end{pmatrix},$$

$$\iota_l(g,t) = \begin{pmatrix} at & -bt & & \\ -bt & dt & & \\ & & at^{-1} & bt^{-1} \\ & & ct^{-1} & dt^{-1} \end{pmatrix},$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. With the latter embedding the linear pair $\mathrm{SL}_2(F) \times \mathrm{GL}_1(F)$ acts by

$$\omega_l(g,t)\varphi(\mathbf{x}) = |t|\varphi\left(g^{-1}t\mathbf{x}\right),$$

since

$${}^*\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\mathrm{t}} \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

The two embeddings are conjugate to each other:

$$T\iota_{os}(g,t)T^{-1} = \iota_l(g,t)$$

where

$$T \coloneqq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Sp}(W_F^0).$$

We denote by $\mathscr{F}$ the operator

$$\begin{aligned} \mathscr{F} \colon \mathscr{S}(\mathbb{A}_F^2) &\longrightarrow \mathscr{S}(\mathbb{A}_F^2) \\ \varphi &\longmapsto \omega(T)\varphi \end{aligned}$$

Using the formula for the Weil representation in (2.1.9) we get

$$\mathscr{F}\varphi\begin{pmatrix} x \\ x' \end{pmatrix} = \int_{\mathbb{A}_F} \varphi\begin{pmatrix} z \\ x' \end{pmatrix} \chi(-xz)dz.$$

It is a partial Fourier transform and satisfies

$$\mathscr{F} \circ \omega_{os}(g,t) = \omega_l(g,t) \circ \mathscr{F}$$

for every $(g,t) \in \mathrm{SL}_2(\mathbb{A}_F) \times \mathbb{A}_F^\times$.

Remark 4.3.3. If $\varphi \begin{pmatrix} x \\ x' \end{pmatrix} = \varphi_1(x)\varphi_2(x')$ with φ_1 and φ_2 in $\mathscr{S}(\mathbb{A}_F)$, then

$$\mathscr{F}\varphi \begin{pmatrix} x \\ x' \end{pmatrix} = \varphi_1^\vee(x)\varphi_2(x')$$

where $\varphi^\vee$ is the Fourier transform on $\mathscr{S}(\mathbb{A}_F)$, see (2.1.4).

For a Schwartz function $\phi \in \mathscr{S}(\mathbb{A}_F^2)$, we define

$$\widetilde{\Theta}_l(g,t,\phi) \coloneqq \sum_{\mathbf{x} \in F^2} \Big(\omega_l(g,t)\phi\Big)(\mathbf{x}) \in C^\infty(F^\times \backslash \mathbb{A}_F^\times)^{K^0(\mathfrak{f})}.$$

If $\mathscr{F}\varphi = \phi$, then by Poisson summation we have

$$\widetilde{\Theta}_l(g,t,\phi) = \widetilde{\Theta}_{os}(g,t,\varphi).$$

Lemma 4.3.11. *Let* $\phi_\infty \coloneqq \mathscr{F}\varphi_\infty \in \mathscr{S}(F_\infty^2)$. *We have*

$$\phi_\infty(\mathbf{x}) = (-i)^N \prod_\sigma e^{-\pi|z_\sigma|^2} z_\sigma$$

where $z_\sigma \coloneqq x_\sigma + i x'_\sigma$.

Proof. We compute

$$\begin{aligned}
\mathscr{F}\varphi_\infty(\mathbf{x}) &= \prod_\sigma \omega(T)\varphi_\sigma \begin{pmatrix} x_\sigma \\ x'_\sigma \end{pmatrix} \\
&= \prod_\sigma \int_{\mathbb{R}} \varphi_\sigma \begin{pmatrix} \alpha_\sigma \\ x'_\sigma \end{pmatrix} e^{-2i\pi\alpha_\sigma x_\sigma} d\alpha_\sigma \\
&= \prod_\sigma \int_{\mathbb{R}} e^{-\pi({x'}_\sigma^2 + \alpha_\sigma^2)} (x'_\sigma + \alpha_\sigma) e^{-2i\pi\alpha_\sigma x_\sigma} d\alpha_\sigma \\
&\overset{u_\sigma = \alpha_\sigma + i x_\sigma}{=} \prod_\sigma e^{-\pi({x'}_\sigma^2 + x_\sigma^2)} \int_{\mathbb{R}} e^{-\pi u_\sigma^2} (u_\sigma + x'_\sigma - i x_\sigma) du_\sigma \\
&= (-i)^N \prod_\sigma e^{-\pi|z_\sigma|^2} z_\sigma.
\end{aligned}$$

$\square$

Theorem 4.3.12. *Suppose that φ_1 or φ_2 vanishes. The diagonal restriction of the Eisenstein series $E(\tau_1 \dots, \tau_N, \phi_f, \psi)$ has the Fourier expansion*

$$E(\tau, \dots, \tau, \phi_f, \psi) = \zeta_f(\varphi_1, \psi^{-1}, 0) + \zeta_f(\varphi_2, \psi, 0) + (-1)^N 2^{N-1} \kappa \sum_{n \in \mathbb{Q}_{>0}} \langle C_n(\varphi_f), C \otimes \psi \rangle e^{2i\pi n\tau},$$

where φ_f is such that $\phi_f = \mathscr{F}\varphi_f$.

Proof. First by Poisson summation we have

$$\begin{aligned} 2^{N-1} I(\tau_1 \dots, \tau_N, \varphi_f, \psi, s) &= (v_1 \cdots v_N)^{-\frac{1}{2}} \int_{F^\times \backslash \mathbb{A}_F^\times} \widetilde{\Theta}_{os}(g_{\underline{\tau}}, t, \varphi) \psi(t) |t|^s dt^\times \\ &\overset{P.S.}{=} (v_1 \cdots v_N)^{-\frac{1}{2}} \int_{F^\times \backslash \mathbb{A}_F^\times} \widetilde{\Theta}_l(g_{\underline{\tau}}, t, \phi) \psi(t) |t|^s dt^\times \\ &= (v_1 \cdots v_N)^{-\frac{1}{2}} \int_{F^\times \backslash \mathbb{A}_F^\times} \sum_{\mathbf{x} \in F^2 - (0,0)} \Big(\omega_l(g_{\underline{\tau}}, t)\phi \Big)(\mathbf{x}) \psi(t) |t|^s dt^\times. \end{aligned}$$

Note that by our choice of Schwartz function ϕ_∞ we have $\phi_\infty(0,0) = 0$, hence the term at $(0,0)$ does not contribute to the summation. We have a bijection

$$\begin{aligned} F^\times \times P(F) \backslash \operatorname{GL}_2(F) &\longrightarrow F^2 - (0,0) \\ (u, \gamma) &\longmapsto u\gamma_0^{-1}\mathbf{x}_0. \end{aligned}$$

where $P(F)$ is the stabilizer of $\mathbf{x}_0 = {}^{\mathrm{t}}(1,0)$ and γ_0 is one of the following representatives in $P(F)$

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \text{ with } \lambda \in F^\times.$$

Hence the sum

$$\int_{F^\times \backslash \mathbb{A}_F^\times} \sum_{\mathbf{x} \in F^2 - (0,0)} \Big(\omega_l(g_{\underline{\tau}}, t)\phi \Big)(\mathbf{x}) \psi(t) |t|^s dt^\times$$

can be unfolded as

$$\begin{aligned} &\int_{F^\times \backslash \mathbb{A}_F^\times} \sum_{u \in F^\times} \sum_{\gamma \in P(F) \backslash \operatorname{GL}_2(F)} \Big(\omega_l(g_{\underline{\tau}}, t)\phi \Big)(u\gamma_0^{-1}\mathbf{x}_0) \psi(t) |t|^s dt^\times \\ &= \int_{\mathbb{A}_F^\times} \sum_{\gamma \in P(F) \backslash \operatorname{GL}_2(F)} \Big(\omega_l(g_{\underline{\tau}}, t)\phi \Big)(u\gamma_0^{-1}\mathbf{x}_0) \psi(t) |t|^s dt^\times. \end{aligned}$$

Since we have termwise absolute convergence for $\mathrm{Re}(s) > N-1$ we can exchange the sum and the integral:

$$\int_{\mathbb{A}_F^\times} \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \Big(\omega_l(g_{\underline{\tau}}, t)\phi\Big)(u\gamma_0^{-1}\mathbf{x}_0)\psi(t)|t|^s dt^\times = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} Z\left(g_{\underline{\tau}}, \gamma_0^{-1}\mathbf{x}_0, \phi, \psi, s\right).$$

Thus we get

$$2^{N-1} I(\tau_1, \ldots, \tau_N, \varphi_f, \psi, s) = E(\tau_1, \ldots, \tau_N, \phi_f, \psi, s),$$

and the Fourier expansion follows from 4.3.7. □

4.4 Classical formulation for quadratic fields

We want to specialize Theorem 4.3.12 to the case where $N = 2$ and $F = \mathbb{Q}(\sqrt{D})$ a quadratic field with $D > 0$ and squarefree. We have $\mathscr{O} = \mathbb{Z}[\lambda]$, where $\lambda := \frac{d_F + \sqrt{d_F}}{2}$. and d_F is the fundamental discriminant. We explicit the choices that allow us to recover [DPV21, Theorem. A].

4.4.1 The symmetric space associated to $\mathrm{SO}(2,2)$

We identify $(F_\mathbb{Q}^2, Q)$ with the quadratic space $(\mathrm{Mat}_2(\mathbb{Q}), 2\det)$ via

$$\begin{aligned} F_\mathbb{Q}^2 &\longrightarrow \mathrm{Mat}_2(\mathbb{Q}) \\ \mathbf{x} = \begin{pmatrix} x \\ x' \end{pmatrix} &\longmapsto [x', SAx], \end{aligned} \tag{4.4.1}$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The fact that this is an isometry follows from $\det[a,b] = {}^{\mathrm{t}}aS^{-1}b$. Let

$$\begin{aligned} \widetilde{H}(\mathbb{Q}) &:= \mathrm{GL}_2(\mathbb{Q}) \times_{\mathbb{Q}^\times} \mathrm{GL}_2(\mathbb{Q}) \\ &= \{(g_1, g_2) \in \mathrm{GL}_2(\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{Q}) \mid \det(g_1) = \det(g_2)\}. \end{aligned}$$

It acts on $\mathrm{Mat}_2(\mathbb{Q})$ by $\tilde{h}X := g_1 X g_2^{-1}$, where $\tilde{h} = (g_1, g_2) \in \widetilde{H}(\mathbb{Q})$. With this identification we have an isomorphism $\widetilde{H} \simeq \mathrm{GSpin}_X$, where GSpin_X is the spin group of the quadratic space $X_\mathbb{Q} = F_\mathbb{Q}^2$. There is an exact sequence

$$1 \longrightarrow \mathbb{Q}^\times \longrightarrow \widetilde{H}(\mathbb{Q}) \xrightarrow{\ \nu\ } H(\mathbb{Q}) \longrightarrow 1$$

where the kernel of ν is given by

$$\mathbb{Q}^\times = \left\{(t\mathbf{1}_2, t\mathbf{1}_2) \in \mathrm{GL}_2(\mathbb{Q})^2 \mid t \in \mathbb{Q}^\times\right\}.$$

Lemma 4.4.1. *The map* $\nu\colon \widetilde{H}(\mathbb{Q}) \longrightarrow H(\mathbb{Q})$ *is given by*

$$\nu\,(g_1, g_2) = \begin{pmatrix} a\,{}^{\#}g_1 & \frac{bA^{-1}Sg_1}{\det(g_2)} \\ \\ cS^{-1}A\,{}^{\#}g_1 & \frac{dg_1}{\det(g_2)} \end{pmatrix},$$

where $g_2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *and* ${}^{\#}g_1 = A^{-1}\,{}^{\mathrm{t}}g_1^{-1}A$.

Proof. We will use several times the equality $-Sg_1S = \det(g_1)\,{}^{\mathrm{t}}g_1^{-1}$. Note that the inverse of the map (4.4.1) is given by

$$[u, u'] \longmapsto \begin{pmatrix} -A^{-1}Su' \\ u \end{pmatrix}.$$

We compute the action separately on the two coordinates $\mathbf{x} = (x, x')$ of $F_{\mathbb{Q}}^2$. First if $x = {}^{\mathrm{t}}(0,0)$:

$$(g_1, g_2) \cdot [x', 0] = \frac{1}{\det(g_2)} g_1[x', 0] \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(g_2)}[dg_1x', -bg_1x']$$

which is mapped in $F_{\mathbb{Q}}^2$ to

$$\frac{1}{\det(g_2)} \begin{pmatrix} bA^{-1}Sg_1x' \\ dg_1x' \end{pmatrix}.$$

On the other hand if $x' = {}^{\mathrm{t}}(0,0)$ we have

$$(g_1, g_2) \cdot [0, SAx] = \frac{1}{\det(g_2)}[0, g_1SAx] \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(g_2)}[-cg_1SAx, ag_1SAx]$$

This is mapped to

$$\begin{pmatrix} a\,{}^{\#}g_1x \\ -cSA\,{}^{\#}g_1x \end{pmatrix} \in F_{\mathbb{Q}}^2.$$

□

The group of real points $\widetilde{H}(\mathbb{R})$ has two connected components and

$$\widetilde{H}(\mathbb{R})^+ = \{(g_1, g_2) \in \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}) \mid \det(g_1) = \det(g_2) > 0\},$$

is the connected component of the identity. Its image is $\nu\left(\widetilde{H}(\mathbb{R})^+\right) = H(\mathbb{R})^+$. We have a transitive action of $\widetilde{H}(\mathbb{R})^+$ on the space $\mathbb{D}^+$ of negative lines in $\mathrm{Mat}_2(\mathbb{R})$ by

$$z \longmapsto \tilde{h}z = g_1 z g_2^{-1},$$

where $\tilde{h} = (g_1, g_2)$. Consider the basis of $\mathrm{Mat}_2(\mathbb{Q})$

$$\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{E}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\mathbf{F}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathbf{F}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the negative plane

$$X_0 = \mathrm{span}\big\langle \mathbf{F}_1, \mathbf{F}_2 \big\rangle = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, a, b \in \mathbb{R} \right\},$$

oriented by $\mathbf{F}_1 \wedge \mathbf{F}_2$. Its stabilizer in $\widetilde{H}(\mathbb{R})^+$ is $\mathbb{R}_{>0}(\mathrm{SO}(2) \times \mathrm{SO}(2))$. Hence the stabilizer of $\alpha X_0 \beta^{-1}$ for $(\alpha, \beta) \in \widetilde{H}(\mathbb{R})^+$ is $\mathbb{R}_{>0}\widetilde{K}_\infty(\alpha X_0 \beta^{-1})$ where

$$\widetilde{K}_\infty(\alpha X_0 \beta^{-1}) = (\alpha, \beta)\, \mathrm{SO}(2) \times \mathrm{SO}(2) (\alpha, \beta)^{-1}.$$

Note that under the isomorphism (4.4.1) the negative plane $z_0 = \{(v, -v), v \in \mathbb{R}^2\}$ is mapped to $g_\infty^{-1} X_0$. On the other hand the group $\widetilde{H}(\mathbb{R})^+$ acts on $\mathbb{H} \times \mathbb{H}$ by $(\tau_1, \tau_2) \mapsto (g_1\tau, g_2\tau_2)$. The stabilizer of $(\alpha i, \beta i)$ is $\mathbb{R}_{>0}\widetilde{K}_\infty(\alpha X_0 \beta^{-1})$. Hence we have isomorphisms

$$\begin{aligned} \mathbb{D}^+ &\longrightarrow \widetilde{H}(\mathbb{R})^+ / \mathbb{R}_{>0}\widetilde{K}_\infty(\alpha X_0 \beta^{-1}) \longrightarrow \mathbb{H} \times \mathbb{H} \\ g_1(\alpha X_0 \beta^{-1}) g_2^{-1} &\longmapsto (g_1, g_2)\mathbb{R}_{>0}\widetilde{K}_\infty(\alpha X_0 \beta^{-1}) \longmapsto (g_1 \alpha i, g_2 \beta i). \end{aligned} \tag{4.4.2}$$

In the other direction we can express the map more concretely by

$$\begin{aligned} \Psi \colon \mathbb{H} \times \mathbb{H} &\longrightarrow \mathbb{D}^+ \\ (\tau_1, \tau_2) &\longmapsto X(\tau_1, \tau_2) = \mathrm{span}\big\langle \mathbf{F}_1(\tau_1, \tau_2), \mathbf{F}_2(\tau_1, \tau_2) \big\rangle \end{aligned} \tag{4.4.3}$$

where

$$\mathbf{F}_1(\tau_1, \tau_2) = \sqrt{y_1 y_2}\, g_{\tau_1} \mathbf{F}_1 g_{\tau_2}^{-1} = \begin{pmatrix} y_1 & -x_2 y_1 - x_1 y_2 \\ 0 & -y_2 \end{pmatrix}$$
$$\mathbf{F}_2(\tau_1, \tau_2) = \sqrt{y_1 y_2}\, g_{\tau_1} \mathbf{F}_2 g_{\tau_2}^{-1} = \begin{pmatrix} x_1 & -x_1 x_2 + y_1 y_2 \\ -1 & x_2 \end{pmatrix}$$

and $g_\tau = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}$ maps i to $\tau = x + iy$.

4.4.2 The adelic isomorphism

Let p be some fixed prime (later we will also assume that p is an odd split prime). Let $\widetilde{K} = \widetilde{K}_\infty(z_0)\widetilde{K}_0(p)$, where

$$\widetilde{K}_0(p) \coloneqq K_0(p) \times_{\det} K_0(p) \subset \widetilde{H}(\widehat{\mathbb{Z}})$$

and

$$K_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}), p \mid c \right\}.$$

By strong approximation for SL_2 we know that

$$\mathrm{SL}_2(\mathbb{A}_f) = \mathrm{SL}_2(\mathbb{Q}) K_0(p).$$

Using the fact that the determinant

$$\det \colon K_0(p)_p \longrightarrow \mathbb{Z}_p^\times$$

is surjective for every p, one can also show that we have $\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2(\mathbb{Q})^+ K_0(p)$. Hence the space

$$M_{\widetilde{K}} = \widetilde{H}(\mathbb{Q}) \backslash \widetilde{H}(\mathbb{A}) / \mathbb{A}^\times \widetilde{K}$$

is connected. The map ν induces an isomorphism

$$M_{\widetilde{K}} \longrightarrow M_K.$$

Since

$$\widetilde{H}(\mathbb{Q})^+ \cap \widetilde{K}_f = \Gamma_0(p) \times \Gamma_0(p)$$

where $\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), p \mid c \right\}$, we have

$$M_{\widetilde{K}} = Y_0(p) \times Y_0(p).$$

Remark 4.4.1. Note that the element $-1 \in H(\mathbb{A})$ correspond to

$$(-\mathbf{1}_2, \mathbf{1}_2) \in \widetilde{H}(\mathbb{Q})^+ \cap \widetilde{K}_0(p),$$

hence in this case we have $\kappa = 2$.

4.4.3 Hecke correspondences

Proposition 4.4.2. *Let $\mathbf{x} \in \mathrm{GL}_2(\mathbb{Q})^+$. After identifying $\mathbb{D}^+$ with $\mathbb{H} \times \mathbb{H}$, the submanifold $\mathbb{D}^+_{\mathbf{x}}$ is the correspondence*

$$\begin{aligned} \mathbb{H} &\hookrightarrow \mathbb{H} \times \mathbb{H} \\ \tau &\longmapsto (\mathbf{x}\tau, \tau). \end{aligned}$$

Proof. We have $g_1 \mathbf{x} g_2^{-1} = \mathbf{x}$ if and only if $g_1 = \mathbf{x} g_2 \mathbf{x}^{-1}$, which means that the stabilizer of $\mathbf{x}$ is

$$\widetilde{H}_{\mathbf{x}}(\mathbb{R})^+ = \left\{ (\mathbf{x} g \mathbf{x}^{-1}, g) \in \widetilde{H}(\mathbb{R})^+ \,\middle|\, g \in \mathrm{GL}_2(\mathbb{R})^+ \right\} \simeq \mathrm{GL}_2(\mathbb{R})^+.$$

We fix the basepoint $\mathbf{x}X_0 \in \mathbb{D}^+$ so that the stabilizer is

$$\widetilde{K}_\infty(\mathbf{x}X_0) \coloneqq \mathbb{R}_{>0}(\mathbf{x}\,\mathrm{SO}(2)\mathbf{x}^{-1} \times \mathrm{SO}(2)).$$

Moreover, the intersection $\widetilde{K}_{\mathbf{x}}(\mathbf{x}X_0) \coloneqq \widetilde{H}_{\mathbf{x}}(\mathbb{R})^+ \cap \widetilde{K}_\infty(\mathbf{x}X_0) \simeq \mathbb{R}_{>0}\,\mathrm{SO}(2)$ and we have an isomorphism

$$\begin{aligned} \mathrm{GL}_2(\mathbb{R})^+/\mathbb{R}_{>0}\,\mathrm{SO}(2) &\longrightarrow \widetilde{H}_{\mathbf{x}}(\mathbb{R})^+/\widetilde{K}_{\mathbf{x}}(\mathbf{x}X_0) \\ g\mathbb{R}_{>0}\,\mathrm{SO}(2) &\longmapsto (\mathbf{x} g \mathbf{x}^{-1}, g)\widetilde{K}_{\mathbf{x}}(\mathbf{x}X_0). \end{aligned}$$

Finally, composing with the identification (4.4.2) we obtain that $\mathbb{D}^+_{\mathbf{x}}$ is the image of

$$\begin{aligned} \mathbb{H} &\longrightarrow \widetilde{H}_{\mathbf{x}}(\mathbb{R})^+/\widetilde{K}_{\mathbf{x}}(\mathbf{x}X_0) \hookrightarrow \mathbb{H} \times \mathbb{H} \\ \tau &\longmapsto (\mathbf{x} g_\tau \mathbf{x}^{-1}, g_\tau)\widetilde{K}_{\mathbf{x}}(\mathbf{x}X_0) \longmapsto (\mathbf{x}\tau, \tau). \end{aligned}$$

□

Let $\Gamma = \Gamma_0(p)$ for some fixed prime p and define the set

$$\Delta_0(p) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \,\middle|\, p \mid c,\ (a,p) = 1,\ ad - bc > 0 \right\},$$

and $\Delta_0(p)^{(n)}$ the subset of matrices of determinant n. Let

$$(\Gamma \times \Gamma)_{\mathbf{x}} = \widetilde{H}_{\mathbf{x}}(\mathbb{R})^+ \cap (\Gamma \times \Gamma) = \left\{ (\mathbf{x}\gamma\mathbf{x}^{-1}, \gamma) \in \Gamma \times \Gamma \,\middle|\, \gamma \in \Gamma_{\mathbf{x}} \right\}$$

where $\Gamma_{\mathbf{x}} := \Gamma \cap \mathbf{x}^{-1}\Gamma\mathbf{x}$. For $\mathbf{x} \in \mathrm{GL}_2(\mathbb{Q})^+$, the special cycle $C_{\mathbf{x}}$ is the correspondence

$$\Gamma_{\mathbf{x}}\backslash\mathbb{H} \hookrightarrow (\Gamma \times \Gamma)_{\mathbf{x}}\backslash\mathbb{H}^2 \longrightarrow Y_0(p) \times Y_0(p)$$

where the first map is $\tau \longmapsto (\mathbf{x}\tau, \tau)$ and the second is the projection

$$(\Gamma \times \Gamma)_{\mathbf{x}}\backslash\mathbb{H}^2 \longrightarrow (\Gamma \times \Gamma)\backslash\mathbb{H}^2 = Y_0(p) \times Y_0(p).$$

Define the open compact

$$\widehat{\Delta}_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\widehat{\mathbb{Z}}) \,\middle|\, a_p \in \mathbb{Z}_p^\times,\ c_p \in p\mathbb{Z}_p \right\},$$

that satisfies $\mathrm{Mat}_2(\mathbb{Q})^+ \cap \widehat{\Delta}_0(p) = \Delta_0(p)$. If we take as Schwartz function

$$\varphi_f = \mathbf{1}_{\widehat{\Delta}_0(p)} \in \mathscr{S}(\mathrm{Mat}_2(\mathbb{A}_F))$$

then for $n > 0$ we have

$$C_n(\mathbf{1}_{\widehat{\Delta}_0(p)}) = \sum_{\mathbf{x} \in \Gamma\backslash\Delta_0(p)^{(n)}/\Gamma} C_{\mathbf{x}}.$$

4.4.4 Orientations

In this section we transfer the orientations defined in Subsection 2.2.3 on $\mathbb{D}^+$ to $\mathbb{H} \times \mathbb{H}$. In order to do so we define a basis of $\mathrm{Mat}_2(\mathbb{Q})$:

$$\begin{aligned}
\mathbf{F}_1(\tau_1, \tau_2) &= \sqrt{y_1 y_2}\, g_{\tau_1} \mathbf{F}_1 g_{\tau_2}^{-1} = \begin{pmatrix} y_1 & -x_2 y_1 - x_1 y_2 \\ 0 & -y_2 \end{pmatrix} \\
\mathbf{F}_2(\tau_1, \tau_2) &= \sqrt{y_1 y_2}\, g_{\tau_1} \mathbf{F}_2 g_{\tau_2}^{-1} = \begin{pmatrix} x_1 & -x_1 x_2 + y_1 y_2 \\ 1 & -x_2 \end{pmatrix} \\
\mathbf{E}_1(\tau_1, \tau_2) &= \sqrt{y_1 y_2}\, g_{\tau_1} \mathbf{E}_1 g_{\tau_2}^{-1} = \begin{pmatrix} y_1 & -x_2 y_1 + x_1 y_2 \\ 0 & y_2 \end{pmatrix} \\
\mathbf{E}_2(\tau_1, \tau_2) &= \sqrt{y_1 y_2}\, g_{\tau_1} \mathbf{E}_2 g_{\tau_2}^{-1} = \begin{pmatrix} x_1 & -x_1 x_2 - y_1 y_2 \\ 1 & -x_2. \end{pmatrix}
\end{aligned}$$

If the basepoint (τ_1, τ_2) is clear we will write $\mathbf{E}_1, \mathbf{E}_2, \mathbf{F}_1, \mathbf{F}_2$. As in Section 2.2.3 we identify

$$T_{X(\tau_1,\tau_2)}\mathbb{D}^+ = \mathrm{Hom}(X(\tau_1,\tau_2), X(\tau_1,\tau_2)^\perp) \simeq X(\tau_1,\tau_2)^\vee \otimes X(\tau_1,\tau_2)^\perp$$

and orient $\mathbb{D}^+$ by

$$(\mathbf{F}_1^\vee \otimes \mathbf{E}_1) \wedge (\mathbf{F}_2^\vee \otimes \mathbf{E}_1) \wedge (\mathbf{F}_1^\vee \otimes \mathbf{E}_2) \wedge (\mathbf{F}_2^\vee \otimes \mathbf{E}_2).$$

Let $\Psi \colon \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{D}^+$ be the isomorphism (4.4.3) and consider the differential

$$d\Psi \colon T_{(\tau_1,\tau_2)}\mathbb{H} \times \mathbb{H} \longrightarrow T_{X(\tau_1,\tau_2)}\mathbb{D}^+.$$

Let $\tau_1 = x_1 + iy_1, \tau_2 = x_2 + iy_2$ be the coordinates on $\mathbb{H} \times \mathbb{H}$.

Proposition 4.4.3. *Under the identification $\mathbb{D}^+ \simeq \mathbb{H} \times \mathbb{H}$ the orientation of $\mathbb{H} \times \mathbb{H}$ is*

$$4y_1^2y_2^2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}$$

and the orientation of the diagonal $\mathbb{D}^+_{\mathbf{1}} \subset \mathbb{D}^+$ at $X(\tau,\tau)$ is given by

$$-y^4 \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \wedge \left(\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_1} \right).$$

Proof.

- Let $\rho_{x_1} \colon (-\epsilon, \epsilon) \longrightarrow \rho_{x_1}(u) = (\tau_1 + u, \tau_2)$ be the path such that $\frac{d}{du}\rho_{x_1}(u)\big|_{u=0} = \frac{\partial}{\partial x_1}$. Note that

$$\begin{aligned} \mathbf{F}_1(\tau_1 + u, \tau_2) &= \mathbf{F}_1(\tau_1, \tau_2) - u \begin{pmatrix} 0 & y_2 \\ 0 & 0 \end{pmatrix} \\ &= \mathbf{F}_1(\tau_1, \tau_2) - \frac{u}{2y_1} \left(\mathbf{F}_2(\tau_1, \tau_2) - \mathbf{E}_2(\tau_1, \tau_2) \right) \end{aligned}$$

and

$$\begin{aligned} \mathbf{F}_2(\tau_1 + u, \tau_2) &= \mathbf{F}_2(\tau_1, \tau_2) + u \begin{pmatrix} 1 & -x_2 \\ 0 & 0 \end{pmatrix} \\ &= \mathbf{F}_2(\tau_1, \tau_2) + \frac{u}{2y_1} \left(\mathbf{F}_1(\tau_1, \tau_2) + \mathbf{E}_1(\tau_1, \tau_2) \right) \end{aligned}$$

Composing $\rho_{x_1}(u)$ with Ψ gives

$$\begin{aligned}\Psi\left(\rho_{x_1}(u)\right) &= \operatorname{span}\big\langle \mathbf{F}_1(\tau_1+u,\tau_2), \mathbf{F}_2(\tau_1+u,\tau_2)\big\rangle \\ &= \operatorname{span}\big\langle \mathbf{F}_1 - \frac{u}{2y_1}\mathbf{F}_2 + \frac{u}{2y_1}\mathbf{E}_2, \mathbf{F}_2 + \frac{u}{2y_1}\mathbf{F}_1 + \frac{u}{2y_1}\mathbf{E}_1\big\rangle \\ &= \operatorname{span}\big\langle 2y_1\mathbf{F}_1 - u\mathbf{F}_2 + u\mathbf{E}_2, 2y_1\mathbf{F}_2 + u\mathbf{F}_1 + u\mathbf{E}_1\big\rangle.\end{aligned}$$

Since $Q(\mathbf{F}_i,\mathbf{F}_j)=0$ and $Q(\mathbf{F}_i)=2$ we have

$$(\alpha\mathbf{F}_1+\beta\mathbf{F}_2)^\vee = \frac{Q(\alpha\mathbf{F}_1+\beta\mathbf{F}_2,-)}{Q(\alpha\mathbf{F}_1+\beta\mathbf{F}_2)} = \frac{\alpha\mathbf{F}_1^\vee+\beta\mathbf{F}_2^\vee}{\alpha^2+\beta^2}.$$

Hence as an element of $X_0^\vee \otimes X_0^\perp$ the plane $\Psi\left(\rho_{x_1}(u)\right)$ is represented by

$$\begin{aligned}&(2y_1\mathbf{F}_1 - u\mathbf{F}_2)^\vee \otimes u\mathbf{E}_2 + (2y_1\mathbf{F}_2 + u\mathbf{F}_1)^\vee \otimes u\mathbf{E}_1 \\ &= \frac{2y_1 u}{4y_1^2+u^2}\left(\mathbf{F}_2^\vee\otimes\mathbf{E}_1 + \mathbf{F}_1^\vee\otimes\mathbf{E}_2\right) + \frac{u^2}{4y_1^2+u^2}\left(\mathbf{F}_1^\vee\otimes\mathbf{E}_1 - \mathbf{F}_2^\vee\otimes\mathbf{E}_2\right).\end{aligned}$$

After taking the derivative at 0 we find that

$$d\Psi\left(\frac{\partial}{\partial x_1}\right) = \frac{1}{2y_1}\left(\mathbf{F}_2^\vee\otimes\mathbf{E}_1 + \mathbf{F}_1^\vee\otimes\mathbf{E}_2\right).$$

- Let $\rho_{y_1}\colon(-\epsilon,\epsilon)\longrightarrow\rho_{y_1}(u)=(\tau_1+iu,\tau_2)$ be the path such that $\frac{d}{du}\rho_{y_1}(u)\big|_{u=0}=\frac{\partial}{\partial y_1}$. Note that

$$\begin{aligned}\mathbf{F}_1(\tau_1+iu,\tau_2) &= \mathbf{F}_1(\tau_1,\tau_2) + u\begin{pmatrix}1 & -x_2\\ 0 & 0\end{pmatrix} \\ &= \mathbf{F}_1(\tau_1,\tau_2) + \frac{u}{2y_1}\left(\mathbf{F}_1(\tau_1,\tau_2)+\mathbf{E}_1(\tau_1,\tau_2)\right)\end{aligned}$$

and

$$\begin{aligned}\mathbf{F}_2(\tau_1+iu,\tau_2) &= \mathbf{F}_2(\tau_1,\tau_2) + u\begin{pmatrix}0 & y_2\\ 0 & 0\end{pmatrix} \\ &= \mathbf{F}_2(\tau_1,\tau_2) + \frac{u}{2y_1}\left(\mathbf{F}_2(\tau_1,\tau_2)-\mathbf{E}_2(\tau_1,\tau_2)\right).\end{aligned}$$

Hence composing $\rho_{y_1}(u)$ with Ψ gives

$$\begin{aligned}\Psi\left(\rho_{y_1}(u)\right) &= \operatorname{span}\left\langle \mathbf{F}_1(\tau_1+iu,\tau_2), \mathbf{F}_2(\tau_1+iu,\tau_2)\right\rangle \\ &= \operatorname{span}\left\langle \left(1+\frac{u}{2y_1}\right)\mathbf{F}_1 + \frac{u}{2y_1}\mathbf{E}_1, \left(1+\frac{u}{2y_1}\right)\mathbf{F}_2 - \frac{u}{2y_1}\mathbf{E}_2\right\rangle \\ &= \operatorname{span}\left\langle (2y_1+u)\,\mathbf{F}_1 + u\mathbf{E}_1, (2y_1+u)\,\mathbf{F}_2 - u\mathbf{E}_2\right\rangle\end{aligned}$$

As an element of $X_0^\vee \otimes X_0^\perp$ this plane is represented by

$$\frac{u}{2y_1+u}\mathbf{F}_1^\vee \otimes \mathbf{E}_1 - \frac{u}{2y_1+u}\mathbf{F}_2^\vee \otimes \mathbf{E}_2.$$

After taking the derivative at 0 we find that

$$d\Psi\left(\frac{\partial}{\partial y_1}\right) = \frac{1}{2y_1}\left(\mathbf{F}_1^\vee \otimes \mathbf{E}_1 - \mathbf{F}_2^\vee \otimes \mathbf{E}_2\right).$$

- Let $\rho_{x_2}\colon (-\epsilon,\epsilon) \longrightarrow \rho_{x_2}(u) = (\tau_1, \tau_2+u)$ be the path such that $\frac{d}{du}\rho_{x_2}(u)\big|_{u=0} = \frac{\partial}{\partial x_2}$. Note that

$$\begin{aligned}\mathbf{F}_1(\tau_1,\tau_2+u) &= \mathbf{F}_1(\tau_1,\tau_2) - u\begin{pmatrix}0 & y_1\\ 0 & 0\end{pmatrix} \\ &= \mathbf{F}_1(\tau_1,\tau_2) - \frac{u}{2y_2}\left(\mathbf{F}_2(\tau_1,\tau_2) - \mathbf{E}_2(\tau_1,\tau_2)\right)\end{aligned}$$

and

$$\begin{aligned}\mathbf{F}_2(\tau_1,\tau_2+u) &= \mathbf{F}_2(\tau_1,\tau_2) - u\begin{pmatrix}0 & x_1\\ 0 & 1\end{pmatrix} \\ &= \mathbf{F}_2(\tau_1,\tau_2) - \frac{u}{2y_2}\left(\mathbf{E}_1(\tau_1,\tau_2) - \mathbf{F}_1(\tau_1,\tau_2)\right).\end{aligned}$$

Composing $\rho_{x_2}(u)$ with Ψ gives

$$\begin{aligned}\Psi\left(\rho_{x_2}(u)\right) &= \operatorname{span}\left\langle \mathbf{F}_1(\tau_1,\tau_2+u), \mathbf{F}_2(\tau_1,\tau_2+u)\right\rangle \\ &= \operatorname{span}\left\langle 2y_2\mathbf{F}_1 - u\mathbf{F}_2 + u\mathbf{E}_2, 2y_1\mathbf{F}_2 + u\mathbf{F}_1 - u\mathbf{E}_1\right\rangle.\end{aligned}$$

Hence as an element of $X_0^\vee \otimes X_0^\perp$ the plane $\Psi\left(\rho_{x_2}(u)\right)$ is represented by

$$\begin{aligned}&(2y_2\mathbf{F}_1 - u\mathbf{F}_2)^\vee \otimes u\mathbf{E}_2 - (2y_2\mathbf{F}_2 + u\mathbf{F}_1)^\vee \otimes u\mathbf{E}_1 \\ &= \frac{2y_2u}{4y_2^2+u^2}\left(\mathbf{F}_1^\vee \otimes \mathbf{E}_2 - \mathbf{F}_2^\vee \otimes \mathbf{E}_1\right) + \frac{u^2}{4y_2^2+u^2}\left(\mathbf{F}_1^\vee \otimes \mathbf{E}_1 + \mathbf{F}_2^\vee \otimes \mathbf{E}_2\right).\end{aligned}$$

After taking the derivative at 0 we find that

$$d\Psi\left(\frac{\partial}{\partial x_2}\right) = \frac{1}{2y_2}\left(\mathbf{F}_1^\vee \otimes \mathbf{E}_2 - \mathbf{F}_2^\vee \otimes \mathbf{E}_1\right).$$

- Let $\rho_{y_2}\colon (-\epsilon,\epsilon) \longrightarrow \rho_{y_1}(u) = (\tau_1, \tau_2 + iu)$ be the path such that $\frac{d}{du}\rho_{y_2}(u)\big|_{u=0} = \frac{\partial}{\partial y_2}$. Note that

$$\begin{aligned}\mathbf{F}_1(\tau_1,\tau_2+iu) &= \mathbf{F}_1(\tau_1,\tau_2) - u\begin{pmatrix}0 & x_1\\ 0 & 1\end{pmatrix}\\ &= \mathbf{F}_1(\tau_1,\tau_2) - \frac{u}{2y_2}\left(\mathbf{E}_1(\tau_1,\tau_2) - \mathbf{F}_1(\tau_1,\tau_2)\right)\end{aligned}$$

and

$$\begin{aligned}\mathbf{F}_2(\tau_1,\tau_2+iu) &= \mathbf{F}_2(\tau_1,\tau_2) + u\begin{pmatrix}0 & y_1\\ 0 & 0\end{pmatrix}\\ &= \mathbf{F}_2(\tau_1,\tau_2) + \frac{u}{2y_2}\left(\mathbf{F}_2(\tau_1,\tau_2) - \mathbf{E}_2(\tau_1,\tau_2)\right).\end{aligned}$$

Hence composing $\rho_{y_2}(u)$ with Ψ gives

$$\begin{aligned}\Psi\left(\rho_{y_2}(u)\right) &= \mathrm{span}\big\langle \mathbf{F}_1(\tau_1,\tau_2+iu), \mathbf{F}_2(\tau_1,\tau_2+iu)\big\rangle\\ &= \mathrm{span}\big\langle (2y_2+u)\,\mathbf{F}_1 - u\mathbf{E}_1, (2y_2+u)\,\mathbf{F}_2 - u\mathbf{E}_2\big\rangle\end{aligned}$$

As an element of $X_0^\vee \otimes X_0^\perp$ this plane is represented by

$$-\frac{u}{2y_2+u}\mathbf{F}_1^\vee \otimes \mathbf{E}_1 - \frac{u}{2y_2+u}\mathbf{F}_2^\vee \otimes \mathbf{E}_2.$$

After taking the derivative at 0 we find that

$$d\Psi\left(\frac{\partial}{\partial y_2}\right) = -\frac{1}{2y_2}\left(\mathbf{F}_1^\vee \otimes \mathbf{E}_1 + \mathbf{F}_2^\vee \otimes \mathbf{E}_2\right).$$

Putting all together we find that

$$\begin{aligned}d\Psi\left(\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}\right) &= \frac{1}{2y_1y_2}(\mathbf{F}_1^\vee \otimes \mathbf{E}_1) \wedge (\mathbf{F}_2^\vee \otimes \mathbf{E}_2)\\ d\Psi\left(\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}\right) &= \frac{1}{2y_1y_2}(\mathbf{F}_2^\vee \otimes \mathbf{E}_1) \wedge (\mathbf{F}_1^\vee \otimes \mathbf{E}_2)\end{aligned}$$

$$
\begin{aligned}
&d\Psi\left(\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1}\wedge\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial y_2}\right)\\
&=\frac{1}{4(y_1y_2)^2}(\mathbf{F}_1^\vee\otimes\mathbf{E}_1)\wedge(\mathbf{F}_2^\vee\otimes\mathbf{E}_1)\wedge(\mathbf{F}_1^\vee\otimes\mathbf{E}_2)\wedge(\mathbf{F}_2^\vee\otimes\mathbf{E}_2).
\end{aligned}
$$

Let us now orient $\mathbb{D}_{\mathbf{1}}^+$ at some point $X(\tau,\tau)\in\mathbb{D}_{\mathbf{1}}^+$. Note that $\mathbf{1}=y^{-1}\mathbf{E}_1$ where $\mathbf{E}_1=\mathbf{E}_1(\tau,\tau)$. As in Subsection 2.2.3 the orientation of the normal bundle is given by

$$o(N_{X(\tau,\tau)}\mathbb{D}_{\mathbf{1}}^+)=(\mathbf{F}_1^\vee\otimes\mathbf{1})\wedge(\mathbf{F}_2^\vee\otimes\mathbf{1})=y^{-2}(\mathbf{F}_1^\vee\otimes\mathbf{E}_1)\wedge(\mathbf{F}_2^\vee\otimes\mathbf{E}_1)$$

and hence

$$o(T_{X(\tau,\tau)}\mathbb{D}_{\mathbf{1}}^+)=y^2(\mathbf{F}_1^\vee\otimes\mathbf{E}_2)\wedge(\mathbf{F}_2^\vee\otimes\mathbf{E}_2).$$

The result follows from

$$
\begin{aligned}
-d\psi\left(y_1\frac{\partial}{\partial y_2}+y_2\frac{\partial}{\partial y_1}\right)&=\mathbf{F}_2^\vee\otimes\mathbf{E}_2\\
d\psi\left(y_1\frac{\partial}{\partial x_1}+y_2\frac{\partial}{\partial x_2}\right)&=\mathbf{F}_1^\vee\otimes\mathbf{E}_2.
\end{aligned}
$$

□

4.4.5 A choice of basis for F

Now suppose that p is an odd split prime in F. Then d_F is a quadratic residue modulo p and we can find $r\in\mathbb{Z}$ such that

$$
\begin{aligned}
r^2&\equiv d_F \pmod p,\\
r^2-d_F&>0.
\end{aligned}
$$

Since the discriminant d_F is always congruent to 0 or 1 modulo 4, it is always a square mod 4 and we can suppose furthermore that

$$r^2\equiv d_F \pmod{4p}.$$

We fix such a root r and set

$$\epsilon_r:=\frac{\sqrt{d_F}-r}{2}\in\mathscr{O}.$$

We take the positive $\mathbb{Z}$-basis $\{\epsilon_r,1\}$ of $\mathscr{O}$. Let $N_0:=2N(\epsilon_r)>0$, so that $2N_0=r^2-d_F$. By changing the root we obtain another basis $\{\epsilon_{-r},1\}$.

Remark 4.4.2. In the previous sections of this chapter, various objects we used implicitely depended on the choice of the $\mathbb{Z}$-basis of $\mathscr{O}$, for example the matrix $g_\infty \in \mathrm{GL}_N(\mathbb{R})$, the embedding $h\colon F_v^\times \longhookrightarrow H(\mathbb{Q}_v)$ or the cycle $C \otimes \psi$. From now we will use the two $\mathbb{Z}$-bases $\{\epsilon_{\pm r}, 1\}$ of $\mathscr{O}$ and decorate the symbols by r, for example we will write $g_{\infty,r}, h_r$ or $C_r \otimes \psi$.

The regular representation with respect to the basis $\{\epsilon_r, 1\}$ is given by

$$\begin{aligned} \gamma_r\colon (F \otimes \mathbb{Q}_v)^\times &\hookrightarrow \mathrm{GL}_2(\mathbb{Q}_v) \\ a\epsilon_r + b &\longmapsto \begin{pmatrix} b - ar & a \\ -\frac{aN_0}{2} & b \end{pmatrix}. \end{aligned} \tag{4.4.4}$$

At all the other split places q we can find $r_{q,r} \in \mathbb{Z}$ such that

$$\begin{aligned} r_{q,r}^2 &\equiv d_F \pmod{q}, \\ r_{q,r}^2 - d_F &> 0, \\ \frac{r_{q,r}}{r} &> 0. \end{aligned} \tag{4.4.5}$$

Let $r_{q,-r} = -r_{q,r}$. As above, for q odd we can suppose furthermore that

$$r_{q,r}^2 \equiv d_F \pmod{4q}.$$

We define

$$\epsilon_{q,r} := \frac{\sqrt{d_F} - r_{q,r}}{2},$$

and since $r_{q,r} \equiv d_F \pmod 2$ we have $\mathscr{O} = \mathbb{Z}[\epsilon_{q,r}]$. Modulo q the minimal polynomial of $\epsilon_{q,r}$ splits as $x(x + r_{q,r})$. We have the splitting $(q) = \mathfrak{q}\mathfrak{q}^\sigma$, where

$$\begin{aligned} \mathfrak{q} &:= \left(\frac{\sqrt{d_F} - r_{q,r}}{2}\right)\mathbb{Z} + q\mathbb{Z} \\ \mathfrak{q}^\sigma &:= \left(\frac{\sqrt{d_F} + r_{q,r}}{2}\right)\mathbb{Z} + q\mathbb{Z}; \end{aligned} \tag{4.4.6}$$

see (2.1.3). At every split prime q we have two square roots of d_F in $\mathbb{Z}_q^\times$. Let $\beta_{q,r} \in \mathbb{Z}_q^\times$ be the one that satisfies

$$\beta_{q,r} \equiv r_{q,r} \pmod{q\mathbb{Z}_q}.$$

Let $\varsigma_{v,r}$ be as in (4.1.4). If v is non-split then $\varsigma_{v,r} = 1$ is the identity on $F_{\mathbb{Q}_v}$, if $v = q$ is split then

$$\begin{aligned} \varsigma_{q,r} \colon F_{\mathbb{Q}_q} &\longrightarrow F_q = \mathbb{Q}_q \times \mathbb{Q}_q \\ \sqrt{d_F} &\longmapsto (\beta_{q,r}, -\beta_{q,r}). \end{aligned}$$

It maps $\mathfrak{q} \otimes \mathbb{Z}_q$ to $(q\mathbb{Z}_q, \mathbb{Z}_q^\times)$ and $\mathfrak{q}^\sigma \otimes \mathbb{Z}_q$ to $(\mathbb{Z}_q^\times, q\mathbb{Z}_q)$. Then, we precompose the map (4.4.4) by the isomorphism $\varsigma_{v,r}^{-1} \colon F_v \longmapsto F_{\mathbb{Q}_v}$, which is the identity at the non split places. This yields an embedding

$$\gamma_r \colon F_v^\times \longhookrightarrow \mathrm{GL}_2(\mathbb{Q}_v)$$

that we also denote by γ_r.

Proposition 4.4.4. *At split primes q the embedding $\gamma_r \colon F_q^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q}_q)$ is defined by*

$$\gamma_r(t_q, t_q^\sigma) = \begin{pmatrix} \frac{t_q + t_q^\sigma}{2} - r\frac{t_q - t_q^\sigma}{2\beta_{q,r}} & \frac{t_q - t_q^\sigma}{\beta_{q,r}} \\ -\frac{(t_q - t_q^\sigma)N_0}{2\beta_{q,r}} & \frac{t_q + t_q^\sigma}{2} + r\frac{t_q - t_q^\sigma}{2\beta_{q,r}} \end{pmatrix}.$$

Proof. By definition $\varsigma_{q,r}$ maps $a\epsilon_r + b$ to $(a\epsilon_r + b, a\epsilon_r^\sigma + b)$, where $\epsilon_r = \frac{\beta_{q,r} - r}{2}, \epsilon_r^\sigma = -\frac{\beta_{q,r} + r}{2}$ are in $\mathbb{Q}_q$. The inverse of $(t_\mathfrak{q}, t_{\mathfrak{q}^\sigma})$ is given by $a\epsilon_r + b \in F_{\mathbb{Q}_q}$ with

$$a = \frac{t_q - t_q^\sigma}{\beta_{q,r}}, \qquad b = \frac{t_q + t_q^\sigma}{2} + r\frac{t_q - t_q^\sigma}{2\beta_{q,r}}$$

and the result follows by applying the map (4.4.4). □

For the basis $\{\epsilon_r, 1\}$ we denote by $h_r \colon F_v^\times \hookrightarrow H(\mathbb{Q}_v)$ the embedding defined in Subsection 4.3.1. Define the map

$$\begin{aligned} \tilde{h}_r \colon F_v^\times &\longhookrightarrow \widetilde{H}(\mathbb{Q}_v) \\ t &\longmapsto \tilde{h}_r(t) = \left(\gamma_r(t), \begin{pmatrix} 1 & 0 \\ 0 & \det(\gamma_r(t)) \end{pmatrix}\right), \end{aligned}$$

which is a lift of the map h_r *i.e.*

$$\nu\left(\tilde{h}_r(t)\right) = h_r(t).$$

At infinity this can be diagonalized as

$$\begin{aligned}&\left(\gamma_r(t_\infty), \begin{pmatrix}1 & 0 \\ 0 & \det(\gamma_r(t_\infty))\end{pmatrix}\right) \\ &= (g_{\infty,r}, 1)^{-1} \left(g(t_\infty), \begin{pmatrix}1 & 0 \\ 0 & \det(g(t_\infty))\end{pmatrix}\right) (g_{\infty,r}, 1),\end{aligned} \tag{4.4.7}$$

where

$$g_{\infty,r} = \begin{pmatrix}\epsilon_r & 1 \\ \epsilon_r^\sigma & 1\end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}).$$

4.4.6 Geodesics on the modular curve and Hecke operators

Suppose we have an embedding

$$\begin{aligned}\Phi\colon \mathbb{R}_{>0} &\longrightarrow \mathbb{H} \\ t &\longmapsto \Phi_x(t) + i\Phi_y(t).\end{aligned} \tag{4.4.8}$$

whose image is the geodesic joining two points $\alpha, \beta \in \mathbb{R} \cup \{\infty\}$. We orient the image $\Phi(\mathbb{R}_{>0})$ by

$$d\Phi\left(\frac{\partial}{\partial t}\right) = \frac{\partial \Phi_x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial \Phi_y}{\partial t}\frac{\partial}{\partial y}.$$

We say that the geodesic is oriented from α to β if

$$\operatorname{sgn} \frac{\partial \Phi_x}{\partial t} = \operatorname{sgn}(\beta - \alpha) \tag{4.4.9}$$

and write $\mathcal{Q}(\alpha, \beta)$ for the image. Note that the condition (4.4.9) is equivalent to $\Phi(0) = \alpha$ and $\Phi(\infty) = \beta$. If α is an RM-point we write $\mathcal{Q}(\alpha) = \mathcal{Q}(\alpha, \alpha^\sigma)$. Let $\overline{\mathcal{Q}}(\alpha, \beta)$ be the image of $\mathcal{Q}(\alpha, \beta)$ in $Y_0(p)$. If α is an RM-point, then $\overline{\mathcal{Q}}(\alpha)$ is a closed (compact) geodesic. Let $-\mathcal{Q}(\alpha) \coloneqq (\alpha^\sigma, \alpha)$.

We define Hecke operators on geodesics as follows. Let $\mathcal{Q} = \mathcal{Q}(\alpha)$ for some RM-point α and

$$\Gamma[\mathcal{Q}] \coloneqq \{\, \gamma \in \Gamma \,|\, \gamma\mathcal{Q} = \mathcal{Q}\}$$

be its stabilizer. It is isomorphic to $\{\pm 1\} \times \gamma_{\mathcal{Q}}^{\mathbb{Z}}$, for some $\gamma_{\mathcal{Q}} \in \Gamma$. In particular, if $P\Gamma$ denotes the image of Γ in $\mathrm{PSL}_2(\mathbb{Z})$ then $P\Gamma[\mathcal{Q}] = \gamma_{\mathcal{Q}}^{\mathbb{Z}}$. Let $R_n \subset \Delta_0(p)^{(n)}$ be a finite set of representatives of $\Gamma\backslash\Delta_0(p)^{(n)}/\Gamma$ such that

$$\Delta_0(p)^{(n)} = \bigsqcup_{\delta \in R_n} \Gamma\delta\Gamma.$$

Let $R_n(\mathcal{Q}) \subset R_n \subset \Delta_0(p)^{(n)}$ be a finite set of representatives of $\Gamma[Q]\backslash\Delta_0(p)^{(n)}/\Gamma$ such that

$$\Delta_0(p)^{(n)} = \bigsqcup_{\delta\in R_n} \Gamma\delta\Gamma = \bigsqcup_{\delta\in R_n(\mathcal{Q})} \Gamma[\mathcal{Q}]\delta\Gamma. \tag{4.4.10}$$

We define the Hecke operator by

$$T_n\overline{\mathcal{Q}} := \sum_{\delta\in R_n(\mathcal{Q})} \overline{\delta^{-1}\mathcal{Q}}.$$

Remark 4.4.3. Consider the set

$$\Delta_0^*(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \middle| p \mid c,\ (a,p)=1,\ ad-bc>0 \right\}.$$

We have a bijection

$$\begin{aligned} \Delta_0(p)^{(n)} &\longrightarrow \Delta_0^*(p)^{(n)} \\ \mathbf{x} &\longrightarrow n\mathbf{x}^{-1}. \end{aligned}$$

If $R_n^*(\mathcal{Q})$ is the image of $R_n(\mathcal{Q})$ then $\Delta_0^*(p)^{(n)} = \bigsqcup_{\delta\in R_n^*(\mathcal{Q})} \Gamma\delta\Gamma[\mathcal{Q}]$ and

$$T_n\overline{\mathcal{Q}} = \sum_{\delta\in R_n^*(\mathcal{Q})} \overline{\delta\mathcal{Q}}.$$

4.4.7 The twisted class $C\otimes\psi$ in $\mathbb{H}\times\mathbb{H}$

Define the point

$$\begin{aligned} \alpha_r &:= g_{\infty,r}^{-1}(0) = -\frac{1}{\epsilon_r} = \frac{r+\sqrt{d_F}}{N_0}, \\ \alpha_{-r} &:= g_{\infty,-r}^{-1}(0) = \frac{-r+\sqrt{d_F}}{N_0} = -\alpha_r^\sigma. \end{aligned}$$

By strong approximation, for every fractional ideal $\mathfrak{a}$ we can find an element $g_{\mathfrak{a},r} \in \mathrm{GL}_2(\mathbb{Q})^+$ such that

$$\gamma_r(t_\mathfrak{a}) \in g_{\mathfrak{a},r}^{-1} K_0(p),$$

where we suppose that $g_{\mathscr{O},r} = 1$. We define the RM-points

$$\alpha_{\mathfrak{a},r} := g_{\mathfrak{a},r}\alpha_r, \qquad \alpha_{\mathfrak{a},-r} := g_{\mathfrak{a},-r}\alpha_{-r}.$$

In particular $\alpha_{\mathscr{O},r} = \alpha_r$.

Remark 4.4.4. Suppose we replace $g_{\mathfrak{a},r}$ by another $\tilde{g}_{\mathfrak{a},r} \in \mathrm{GL}_2(\mathbb{Q})^+$ that satisfies

$$\gamma_r(t_{\mathfrak{a}}) \in \tilde{g}_{\mathfrak{a},r}^{-1} K_0(p),$$

and let $\widetilde{\alpha}_{\mathfrak{a},r} = \tilde{g}_{\mathfrak{a},r}\alpha_r$. Then

$$\tilde{g}_{\mathfrak{a},r} g_{\mathfrak{a},r}^{-1} \in \mathrm{GL}_2(\mathbb{Q})^+ \cap K_0(p) = \Gamma$$

and hence $\widetilde{\alpha}_{\mathfrak{a},r} \in \Gamma\alpha_{\mathfrak{a},r}$.

We suppose that $h^+ = 2h$, since $h^+ = h$ is equivalent to the existence of a unit of negative norm, which implies that the Hecke character is not totally odd, see (2.1.8) and (2.1.5). Hence we have $\mathrm{Cl}(F)^+ = \mathrm{Cl}(F) \sqcup \mathfrak{c}\mathrm{Cl}(F)$ where $\mathfrak{c} = (\sqrt{d_F})$.

Lemma 4.4.5. *The orbit $\Gamma\alpha_{\mathfrak{a},r}$ only depends on the narrow class $[\mathfrak{a}] \in Cl(F)^+$. Moreover we have:*

1. $\Gamma\alpha_{\mathfrak{ca},r} = \Gamma(-\alpha_{\mathfrak{a},r})$,
2. $\Gamma\alpha_{\mathfrak{a}^\sigma,r} = \Gamma(-\alpha_{\mathfrak{a},r}^\sigma)$,
3. $\Gamma\alpha_{\mathfrak{a},-r} = \Gamma(-\alpha_{\mathfrak{a},r}^\sigma)$.

In particular $\overline{\mathcal{Q}}(\alpha_{\mathfrak{a},r})$ only depends on the narrow class of $\mathfrak{a}$, and we have

$$\overline{\mathcal{Q}}(\alpha_{\mathfrak{a},-r}) = \overline{\mathcal{Q}}(\alpha_{\mathfrak{a}^\sigma,r}), \qquad \overline{\mathcal{Q}}(\alpha_{\mathfrak{ca},-r}) = \overline{\mathcal{Q}}(\alpha_{\mathfrak{ca}^\sigma,r}).$$

Proof. First note that if $\lambda \in F^\times$, then

$$\gamma_r(\lambda)\alpha_r = \alpha_r,$$

since

$$\gamma_r(\lambda) = g_{\infty,r}^{-1} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^\sigma \end{pmatrix} g_{\infty,r}.$$

Now suppose that we replace $\mathfrak{a}$ by $\mathfrak{b} = (\lambda)\mathfrak{a}$ and λ totally positive. Then we have

$$\gamma_r(t_{\mathfrak{b}}) \in \gamma_r(\lambda) g_{\mathfrak{a},r}^{-1} K_0(p)$$

hence by the previous remark we have for $g_{\mathfrak{b},r} = g_{\mathfrak{a},r}\gamma_r(\lambda)^{-1}$

$$\alpha_{\mathfrak{b},r} \in \Gamma g_{\mathfrak{a},r}\gamma_r(\lambda)^{-1}\alpha_r = \Gamma\alpha_{\mathfrak{a},r}.$$

1. If $\mathfrak{b} = \mathfrak{c}\mathfrak{a}$ where $\mathfrak{c} = (\sqrt{d_F})$. We have
$$\gamma_r(t_{\mathfrak{c}}) = \gamma_r(\sqrt{d_F}) \in \mathrm{GL}_2(\mathbb{Q})^-$$
and fixes α_r. Then $g_{\mathfrak{a},r}\gamma_r(\sqrt{d_F})^{-1} \in \mathrm{GL}_2(\mathbb{Q})^-$ and we have
$$g_{\mathfrak{a},r}\gamma_r(\sqrt{d_F})^{-1}\gamma_r(t_{\mathfrak{c}\mathfrak{a}})^{-1} = g_{\mathfrak{a},r}\gamma_r(t_{\mathfrak{a}})^{-1} \in K_0(p).$$
Finally we multiply by $\delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}) \cap K_0(p)$ so that $\delta g_{\mathfrak{a},r}\gamma_r(\sqrt{d_F})^{-1}$ has positive determinant. By Remark 4.4.4 we have for $g_{\mathfrak{c}\mathfrak{a},r} = \delta g_{\mathfrak{a},r}\gamma_r(\sqrt{d_F})^{-1}$
$$\Gamma\alpha_{\mathfrak{c}\mathfrak{a},r} = \Gamma\delta\alpha_{\mathfrak{a},r} = \Gamma(-\alpha_{\mathfrak{a},r}).$$

2. Replacing $\mathfrak{a}$ by $\mathfrak{a}^\sigma$ amounts to precompose the map in Proposition 4.4.4 by the involution $(t_q, t_q^\sigma) \mapsto (t_q^\sigma, t_q)$ at split primes. A direct computation shows that $\gamma_r(t_{\mathfrak{a}^\sigma}) = M^{-1}\gamma_r(t_{\mathfrak{a}})M$ where
$$M = \begin{pmatrix} 0 & 2 \\ N_0 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})^- \cap K_0(p).$$
Again after multiplying by δ we have that $\delta g_{\mathfrak{a},r}M\gamma_r(t_{\mathfrak{a}^\sigma}) \in K_0(p)$. We have $M\alpha_r = \alpha_r^\sigma$, hence
$$\Gamma\alpha_{\mathfrak{a}^\sigma,r} = \Gamma\delta g_{\mathfrak{a},r}M\alpha_r = \Gamma(-\alpha_{\mathfrak{a},r}^\sigma).$$

3. Note that $\beta_{q,-r} = -\beta_{q,r}$. Hence we have $\gamma_{-r}(t_{\mathfrak{a}}) = \delta\gamma_r(t_{\mathfrak{a}})\delta$, and
$$\Gamma\alpha_{\mathfrak{a},-r} = \Gamma g_{\mathfrak{a},-r}\alpha_{-r} = \Gamma\delta g_{\mathfrak{a},r}\delta\alpha_{-r} = \Gamma\delta g_{\mathfrak{a},r}\alpha_r^\sigma = \Gamma(-\alpha_{\mathfrak{a},r}^\sigma)$$

□

Suppose that ψ is unramified, so $\mathfrak{f} = \mathscr{O}$. Let $K^0(\mathscr{O}) = K_\infty^0 \times \widehat{\mathscr{O}}^\times$ where
$$K_\infty^0 = \left\{ (t_1, t_2) \in F_\infty^\times \;\middle|\; t_i = \pm 1, t_1 t_2 = 1 \right\}.$$

We have
$$M_{\mathscr{O}} = F^\times \backslash \mathbb{A}_F^\times / K^0(\mathscr{O}) = \bigsqcup_{\mathfrak{a} \in \mathrm{Cl}(F)^+} \widehat{\mathscr{O}}^{\times,+} \backslash \mathbb{R}_{>0}^2,$$

where $\mathrm{Cl}(F)^+$ is the narrow class group.

Lemma 4.4.6. *We have* $\tilde{h}_r(\widehat{\mathscr{O}}^\times) \subset \widetilde{K}_0(p)$ *and* $\tilde{h}_r(K^0_\infty) = \tilde{h}_r((\mathbb{R}^\times)^N) \cap \widetilde{K}_\infty(z_0)$.

Proof. Since $\mathrm{N}(\epsilon_r) \in p\mathbb{Z}_p$ and $2\beta_q \in \mathbb{Z}_q^\times$ for all split odd primes q, it follows that for the embedding in Proposition 4.4.4

$$\gamma_r(\mathscr{O}_q^\times) \subset K_0(p)_q. \tag{4.4.11}$$

Suppose that $q = 2$ is split in F, and let $t_2, t_2^\sigma \in \mathbb{Z}_2^\times$. Then $t_2 \equiv t_2^\sigma \equiv 1 \pmod{2\mathbb{Z}_2}$, hence

$$\left|\frac{t_2 + t_2^\sigma}{2}\right|_2 \leq 1, \qquad \left|\frac{t_2 - t_2^\sigma}{2}\right|_2 \leq 1,$$

and (4.4.11) also holds at $q = 2$. Then, since $\det(\gamma_r(t_q, t_q^\sigma)) = t_q t_q^\sigma$ is in $\mathbb{Z}_q^\times$ for $t_q, t_q^\sigma \in \mathbb{Z}_q^\times$, we have

$$\tilde{h}_r(\mathscr{O}_q^\times) \subset \widetilde{K}_0(p)_q.$$

On the other hand at the non split primes q we have $\mathscr{O}_q \simeq \mathscr{O} \otimes \mathbb{Z}_q$, and the fact that $\gamma_r(\mathscr{O}_q^\times) \subset \mathrm{GL}_2(\mathbb{Z}_q)$ follows from the fact that $\{\epsilon_r, 1\}$ is a $\mathbb{Z}$-basis of $\mathscr{O}$. Finally, since $z_0 = g_{\infty,r}^{-1} X_0$ we have

$$\widetilde{K}_\infty(z_0) = (g_{\infty,r}^{-1}, 1)\,\mathrm{SO}(2) \times \mathrm{SO}(2)(g_{\infty,r}, 1),$$

and the second statement follows by (4.4.7). □

It follows from the lemma that the embedding $\tilde{h}_r$ induces a map

$$\tilde{h}_r \colon M_{\mathscr{O}} \longrightarrow M_{\widetilde{K}}$$

whose image in $M_{\widetilde{K}}$ corresponds to the image of h_r in $M_K \simeq M_{\widetilde{K}} \simeq Y_0(p) \times Y_0(p)$. Let $C_{\mathfrak{a},r} \in \mathcal{Z}_2(\overline{M_K}, \partial \overline{M_K}; \mathbb{R})$ be the image of a connected component of $\mathscr{O}^{\times,+} \backslash \mathbb{R}^2_{>0}$ by h_r, that we considered in Subsection 4.3.2. Similarly, we define

$$C_r \otimes \psi = \sum_{[\mathfrak{a}] \in \mathrm{Cl}(F)^+} \psi(\mathfrak{a}) C_{\mathfrak{a},r} \in \mathcal{Z}_2(\overline{M_K}, \partial \overline{M_K}; \mathbb{R}).$$

Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be real numbers. There is a natural product orientation on $\mathcal{Q}(\alpha_1, \beta_1) \times \mathcal{Q}(\alpha_2, \beta_2) \subset \mathbb{H} \times \mathbb{H}$. If Φ_1, Φ_2 are diffeomorphisms as in (4.4.8). Then we orient the product by

$$\begin{aligned} &d(\Phi_1 \times \Phi_2)\left(t_1 \frac{\partial}{\partial t_1} \wedge t_2 \frac{\partial}{\partial t_2}\right) \\ &= t_1 t_2 \left(\frac{\partial \Phi_{1,x}}{\partial t_1} \frac{\partial}{\partial x_1} + \frac{\partial \Phi_{1,y}}{\partial t_1} \frac{\partial}{\partial y_1}\right) \wedge \left(\frac{\partial \Phi_{2,x}}{\partial t_2} \frac{\partial}{\partial x_2} + \frac{\partial \Phi_{2,y}}{\partial t_2} \frac{\partial}{\partial y_2}\right). \end{aligned}$$

Proposition 4.4.7. *After identifying M_K with $Y_0(p) \times Y_0(p)$ we have*

$$C_{\mathfrak{a},r} = \overline{\mathcal{Q}}(\alpha_{\mathfrak{a},r}) \times \overline{\mathcal{Q}}(\infty, 0).$$

Proof. Let $\mathscr{O}^{\times,+}t_\infty$ be in the connected component of $M_{\mathscr{O}}$ corresponding to $\mathfrak{a}$. It corresponds to the point $F^\times(t_\infty, t_{\mathfrak{a}})K^0(\mathscr{O}) \in M_{\mathscr{O}}$, which is mapped to $\widetilde{H}(\mathbb{Q})(\tilde{h}_r(t_\infty), \tilde{h}_r(t_{\mathfrak{a}}))\widetilde{K}$. By definition

$$\tilde{h}_r(t_\infty) = (g_{\infty,r}^{-1}, 1)\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & t_1t_2 \end{pmatrix}\right)(g_{\infty,r}, 1),$$

where $g_{\infty,r} = \begin{pmatrix} \epsilon_r & 1 \\ \epsilon_r^\sigma & 1 \end{pmatrix}$ and the orientation is given by $t_1\frac{\partial}{\partial t_1} \wedge t_2\frac{\partial}{\partial t_2}$. The image in $\mathbb{H} \times \mathbb{H}$ is the set

$$\begin{aligned} \mathbb{D}_0^+ &= \left\{\left(g_{\infty,r}^{-1}\frac{t_1}{t_2}i, \frac{1}{t_1t_2}i\right), t_1, t_2 > 0\right\} \\ &= \left\{\left(g_{\infty,r}^{-1}u_1 i, \frac{1}{u_2}i\right), u_1, u_2 > 0\right\} \end{aligned}$$

after the change of variable $(u_1, u_2) = \left(\frac{t_1}{t_2}, t_1t_2\right)$. It preserves the orientation since

$$t_1\frac{\partial}{\partial t_1} \wedge t_2\frac{\partial}{\partial t_2} = 2u_1\frac{\partial}{\partial u_1} \wedge u_2\frac{\partial}{\partial u_2}.$$

We can parametrize $\mathbb{D}_0^+$ by

$$\begin{aligned} \mathbb{R}_{>0}^2 &\longrightarrow \mathbb{H} \times \mathbb{H} \\ (u_1, u_2) &\longmapsto (\Phi_1(u_1), \Phi_2(u_2)) \end{aligned}$$

where

$$\begin{aligned} \Phi_1(u_1) &= g_{\infty,r}^{-1}u_1 i = -\frac{\epsilon_r + u_1^2\epsilon_r^\sigma}{\epsilon_r^2 + u_1^2(\epsilon_r^\sigma)^2} + i\frac{u_1(\epsilon_r - \epsilon_r^\sigma)}{\epsilon_r^2 + u_1^2(\epsilon_r^\sigma)^2} \\ \Phi_2(u_2) &= \frac{1}{u_2}i. \end{aligned}$$

Since $\Phi_1(0) = -\frac{1}{\epsilon_r} = \alpha_r$ and $\Phi_1(\infty) = -\frac{1}{\epsilon_r^\sigma} = \alpha_r^\sigma < \Phi_1(0)$ we find that

$$\mathbb{D}_0^+ = \mathcal{Q}(\alpha_r) \times \mathcal{Q}(\infty, 0), \tag{4.4.12}$$

at least without the orientations. The orientation at $(\Phi_1(u_1), \Phi_2(u_2)) \in \mathcal{Q}(\alpha_r) \times \mathcal{Q}(\infty, 0)$ is given by

$$2\left(u_1\frac{\partial\Phi_{1,x}}{\partial u_1}\frac{\partial}{\partial x_1}+u_1\frac{\partial\Phi_{1,y}}{\partial u_1}\frac{\partial}{\partial y_1}\right)\wedge u_2\frac{\partial\Phi_{2,y}}{\partial u_2}\frac{\partial}{\partial y_2}$$

where

$$\begin{aligned}\frac{\partial\Phi_{1,x}}{\partial u_1}&=-\sqrt{d_F}\frac{2u_1\,\mathrm{N}(\epsilon_r)}{\left(\epsilon_r^2+u_1^2(\epsilon_r^\sigma)^2\right)^2}<0\\ \frac{\partial\Phi_{1,y}}{\partial u_1}&=\sqrt{d_F}\frac{\epsilon_r^2-u_1^2(\epsilon_r^\sigma)^2}{\left(\epsilon_r^2+u_1^2(\epsilon_r^\sigma)^2\right)^2}\\ \frac{\partial\Phi_{2,y}}{\partial u_2}&=-\frac{1}{u_2^2}<0.\end{aligned}$$

Hence the orientation on both sides of (4.4.12) match.

We write $\tilde{h}_r(t_{\mathfrak{a}})=\tilde{h}_{\mathfrak{a},r}^{-1}\tilde{k}_f\in\widetilde{H}(\mathbb{Q})^+\widetilde{K}_f$, with

$$\tilde{h}_{\mathfrak{a},r}=\left(g_{\mathfrak{a},r},\begin{pmatrix}1&0\\0&\det(g_{\mathfrak{a},r})\end{pmatrix}\right).$$

Since $g_{\mathfrak{a},r}\mathcal{Q}(\alpha_r)=\mathcal{Q}(\alpha_{\mathfrak{a},r})$ it follows that

$$\mathbb{D}_{\mathfrak{a}}^+=\mathcal{Q}(\alpha_{\mathfrak{a},r})\times\mathcal{Q}(\infty,0),$$

and that $C_{\mathfrak{a},r}$ is the image in $Y_0(p)^2$ of $\mathcal{Q}(\alpha_{\mathfrak{a},r})\times\mathcal{Q}(\infty,0)$. □

We set

$$\overline{\mathcal{Q}}(\psi):=\sum_{[\mathfrak{a}]\in\mathrm{Cl}(F)^+}\psi(\mathfrak{a})\left(\overline{\mathcal{Q}}(\alpha_{\mathfrak{a},r})+\overline{\mathcal{Q}}(\alpha_{\mathfrak{a},-r})\right)$$

so that

$$\overline{\mathcal{Q}}(\psi)\times\overline{\mathcal{Q}}(\infty,0)=C_r\otimes\psi+C_{-r}\otimes\psi.$$

Remark 4.4.5. By using the involution $[\mathfrak{a}]\mapsto[\mathfrak{a}^\sigma]=[\mathfrak{a}]^{-1}$ and the fact that $\alpha_{\mathfrak{a}^\sigma,r}$ and $\alpha_{\mathfrak{a},-r}$ are in the same Γ-orbit we find that $\overline{\mathcal{Q}}(\psi)=\overline{\mathcal{Q}}(\psi^{-1})$.

4.4.8 Intersection numbers of geodesics

Let $\mathcal{Q}(\alpha_1,\beta_1)$ and $\mathcal{Q}(\alpha_2,\beta_2)$ be two geodesics with pairwise distinct endpoints. We fix the orientation $y^2\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$ on $\mathbb{H}$ and define the intersection of $\mathcal{Q}(\alpha_1,\beta_1)$ and $\mathcal{Q}(\alpha_2,\beta_2)$ as in Subsection 2.1.1. In particular, the intersection with the winding element $\mathcal{Q}(0,\infty)$ is

$$\langle \mathcal{Q}(0,\infty), \mathcal{Q}(\alpha,\beta)\rangle = \begin{cases} 1 & \text{if } \beta < 0 < \alpha \\ -1 & \text{if } \alpha < 0 < \beta \\ 0 & \text{else,} \end{cases}$$

see Figure 4.1 below. Since the geodesics are 1 dimensional we have

$$\langle \mathcal{Q}(\alpha_1,\beta_1), \mathcal{Q}(\alpha_2,\beta_2)\rangle = -\langle \mathcal{Q}(\alpha_2,\beta_2), \mathcal{Q}(\alpha_1,\beta_1)\rangle = \langle \mathcal{Q}(\beta_2,\alpha_2), \mathcal{Q}(\alpha_1,\beta_1)\rangle.$$

If $\mathcal{Q} = \mathcal{Q}(\alpha)$ for some RM-point α we can define the intersection on $Y_0(p)$ by

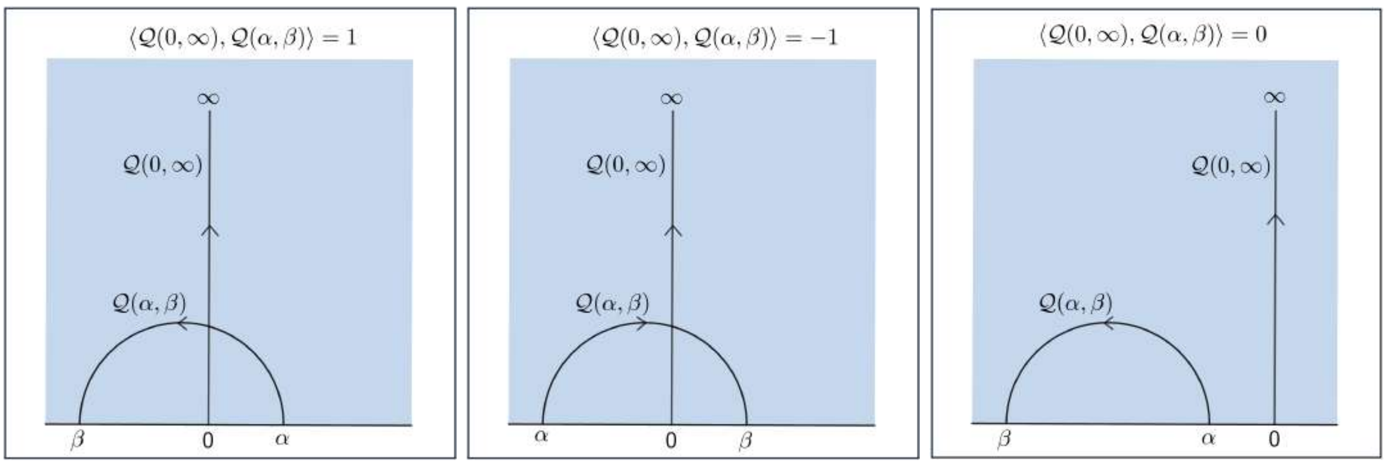


Figure 4.1: Intersection of geodesics

$$\langle \overline{\mathcal{Q}}(0,\infty), \overline{\mathcal{Q}}\rangle := \sum_{\gamma\in\Gamma/\Gamma[\mathcal{Q}]} \langle \mathcal{Q}(0,\infty), \gamma\mathcal{Q}\rangle. \tag{4.4.13}$$

Let $\tau \in \mathcal{Q}$ be any point on $\mathcal{Q}$ and $[\tau, \gamma_{\mathcal{Q}}\tau)$ the half-open geodesic segment, where γ_Q is a generator of $\Gamma[Q] = \{\pm 1\} \times \gamma_Q^{\mathbb{Z}}$. Then

$$\mathcal{Q} = \bigsqcup_{i\in\mathbb{Z}} [\gamma_{\mathcal{Q}}^i\tau, \gamma_{\mathcal{Q}}^{i+1}\tau) \tag{4.4.14}$$

and we also have

$$\langle \overline{\mathcal{Q}}(0,\infty), \overline{\mathcal{Q}}\rangle = \sum_{\gamma\in P\Gamma} \langle \mathcal{Q}(0,\infty), [\gamma\tau, \gamma\gamma_{\mathcal{Q}}\tau)\rangle. \tag{4.4.15}$$

The intersection on the right hand side is only-non zero for finitely many γ's.

Lemma 4.4.8. *Let α and β be real numbers and $\mathbf{y} \in \mathrm{Mat}_2(\mathbb{Z})$ with $\det(\mathbf{y}) > 0$. Then*

$$\left\langle \mathbb{D}^+_{\mathbf{y}}, \mathcal{Q}(\alpha,\beta) \times \mathcal{Q}(\infty, 0)\right\rangle = \left\langle \mathcal{Q}(0,\infty), \mathbf{y}^{-1}\mathcal{Q}(\alpha,\beta)\right\rangle.$$

Proof. First since $\mathbb{D}^+_{\mathbf{y}} = \mathbf{y}\mathbb{D}^+_{\mathbf{1}}$ we have

$$\begin{aligned}\left\langle \mathbb{D}^+_{\mathbf{y}}, \mathcal{Q}(\alpha,\beta) \times \mathcal{Q}(\infty, 0)\right\rangle &= \left\langle \mathbb{D}^+_{\mathbf{1}}, \mathbf{y}^{-1}\mathcal{Q}(\alpha,\beta) \times \mathcal{Q}(\infty, 0)\right\rangle \\ &= \left\langle \mathbb{D}^+_{\mathbf{1}}, \mathcal{Q}(\mathbf{y}^{-1}\alpha,\mathbf{y}^{-1}\beta) \times \mathcal{Q}(\infty, 0)\right\rangle\end{aligned}$$

where $\mathbb{D}^+_{\mathbf{1}}$ is the diagonal embedding of $\mathbb{H}$ in $\mathbb{H}\times\mathbb{H}$. We set $\alpha' = \mathbf{y}^{-1}\alpha$ and $\beta' = \mathbf{y}^{-1}\beta$. We have a bijection

$$\begin{aligned}\mathcal{Q}(\alpha',\beta') \cap \mathcal{Q}(\infty,0) &\longrightarrow (\mathcal{Q}(\alpha',\beta') \times \mathcal{Q}(\infty,0)) \cap \mathbb{D}^+_{\mathbf{1}} \\ \tau &\longmapsto (\tau,\tau),\end{aligned}$$

and these intersections contain at most one point.

We only have to check that the signs of the orientations matches when the intersection is non-empty. Let $\tau \in \mathcal{Q}(0,\infty)\cap\mathcal{Q}(\alpha',\beta')$, and suppose that $\alpha' < 0 < \beta'$. Then

$$\left\langle \mathcal{Q}(0,\infty), \mathcal{Q}(\alpha',\beta')\right\rangle = -1. \tag{4.4.16}$$

At the point τ the orientation on $\mathcal{Q}(\alpha',\beta')$ is given by $a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial y_1}$ with $a > 0$, and on $\mathcal{Q}(\infty,0)$ by $-\frac{\partial}{\partial y_2}$. Hence the orientation on $\mathcal{Q}(\alpha',\beta')\times\mathcal{Q}(\infty,0)$ is given by $-\left(a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial y_1}\right)\wedge\frac{\partial}{\partial y_2}$. By Proposition 4.4.3 the orientation of $\mathbb{D}^+_{\mathbf{1}}$ is given (up to a positive scalar) by $-\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\wedge\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right)$. Since

$$\begin{aligned}&o(T_{X(\tau,\tau)}\mathbb{D}^+_{\mathbf{1}}) \wedge o(T_{X(\tau,\tau)}\mathcal{Q}(\alpha,\beta)\times\mathcal{Q}(\infty,0)) \\ &= \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\wedge\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right)\wedge\left(a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial y_1}\right)\wedge\frac{\partial}{\partial y_2} \\ &= a\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial y_1}\wedge\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_2} \\ &= -a\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1}\wedge\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial y_2}\end{aligned}$$

and $a > 0$ we see that

$$\left\langle \mathbb{D}^+_{\mathbf{1}}, \mathcal{Q}(\alpha',\beta')\times\mathcal{Q}(\infty,0)\right\rangle = -1. \tag{4.4.17}$$

When $\alpha' < 0 < \beta'$ then $a < 0$ and (4.4.16),(4.4.17) are both 1.

□

Proposition 4.4.9. *For* $\varphi_f = \mathbf{1}_{\widehat{\Delta}_0(p)}$ *we have*

$$\int_{\overline{\mathcal{Q}}(\alpha_{\mathfrak{a},r})\times\overline{\mathcal{Q}}(\infty,0)} \Theta_n(v,\varphi_f) = 2\big\langle \overline{\mathcal{Q}}(0,\infty), T_n\overline{\mathcal{Q}}(\alpha_{\mathfrak{a},r})\big\rangle.$$

Proof. By definition, for $\varphi_f = \mathbf{1}_{\widehat{\Delta}_0(p)}$ we have

$$\Theta_n(v,\varphi_f) = \sum_{\substack{\mathbf{x}\in \mathrm{Mat}_2(\mathbb{Q})\\ Q(\mathbf{x},\mathbf{x})=2n}} \varphi_f(\mathbf{x})\varphi^0(\sqrt{v}\mathbf{x}) = \sum_{\mathbf{x}\in\Delta_0(p)^{(n)}} \varphi^0(\sqrt{v}\mathbf{x}).$$

We write $\mathcal{Q}=\mathcal{Q}(\alpha_{\mathfrak{a},r})$. Let $\tau\in\mathcal{Q}$ be any point on $\mathcal{Q}$ and $[\tau,\gamma_{\mathcal{Q}}\tau)$ the half-open geodesic segment. The group $\Gamma[\mathcal{Q}]\times\big(\Gamma/\Gamma[\delta^{-1}\mathcal{Q}]\big)$ acts transitively on $\Gamma[\mathcal{Q}]\delta\Gamma$, where $\Gamma[\delta^{-1}\mathcal{Q}]=\delta^{-1}\Gamma[\mathcal{Q}]\delta$. Using the decomposition (4.4.10) we write

$$\begin{aligned}\Theta_n(v,\varphi_f) &= \sum_{\mathbf{x}\in\Delta_0(p)^{(n)}} \varphi^0(\sqrt{v}\mathbf{x})\\ &= \sum_{\delta\in R_n(\mathcal{Q})}\sum_{\gamma_1\in\Gamma[\mathcal{Q}]}\sum_{\gamma_2\in\Gamma/\Gamma[\delta^{-1}\mathcal{Q}]} \varphi^0(\sqrt{v}\gamma_1\delta\gamma_2^{-1}).\end{aligned}$$

We get

$$\int_{\overline{\mathcal{Q}}\times\overline{\mathcal{Q}}(\infty,0)} \Theta_n(v,\varphi_f) = \sum_{\delta\in R_n(\mathcal{Q})}\sum_{\gamma_1\in\Gamma[\mathcal{Q}]}\sum_{\gamma_2\in\Gamma/\Gamma[\delta^{-1}\mathcal{Q}]}\int_{[\tau,\gamma_{\mathcal{Q}}\tau)\times\mathcal{Q}(\infty,0)} \varphi^0(\sqrt{v}\gamma_1\delta\gamma_2^{-1}).$$

By the invariance of φ^0 we have $\varphi^0(\sqrt{v}\gamma_1\delta\gamma_2^{-1}) = (\gamma_1^{-1},1)^*\varphi^0(\sqrt{v}\delta\gamma_2^{-1})$. Thus using (4.4.14) we get

$$\begin{aligned}\sum_{\gamma_1\in\Gamma[\mathcal{Q}]}\int_{[\tau,\gamma_{\mathcal{Q}}\tau)\times\mathcal{Q}(\infty,0)} \varphi^0(\sqrt{v}\gamma_1\delta\gamma_2^{-1}) &= \sum_{\gamma_1\in\Gamma[\mathcal{Q}]}\int_{[\tau,\gamma_{\mathcal{Q}}\tau)\times\mathcal{Q}(\infty,0)} (\gamma_1^{-1},1)^*\varphi^0(\sqrt{v}\delta\gamma_2^{-1})\\ &= \sum_{\gamma_1\in\Gamma[\mathcal{Q}]}\int_{[\gamma_1^{-1}\tau,\gamma_1^{-1}\gamma_{\mathcal{Q}}\tau)\times\mathcal{Q}(\infty,0)} \varphi^0(\sqrt{v}\delta\gamma_2^{-1})\\ &= 2\int_{\mathcal{Q}\times\mathcal{Q}(\infty,0)} \varphi^0(\sqrt{v}\delta\gamma_2^{-1}),\end{aligned}$$

and the factor 2 comes from the subgroup $\{\pm1\}$ of $\Gamma[\mathcal{Q}]$. Since $\mathbb{D}^+_{\mathfrak{a}} = \mathcal{Q}(\alpha_{\mathfrak{a},r})\times\mathcal{Q}(\infty,0)$ it follows from (4.3.17) in the proof of Proposition 4.3.10 that

$$\int_{\mathcal{Q}\times\mathcal{Q}(\infty,0)} \varphi^0(\sqrt{v}\delta\gamma_2^{-1}) = \big\langle \mathbb{D}^+_{\delta\gamma_2^{-1}}, \mathcal{Q}\times\mathcal{Q}(\infty,0)\big\rangle,$$

and by Lemma 4.4.8 we have

$$\left\langle \mathbb{D}^{+}_{\delta\gamma_2^{-1}}, \mathcal{Q} \times \mathcal{Q}(\infty, 0) \right\rangle = \left\langle \mathcal{Q}(0, \infty), \gamma_2 \delta^{-1} \mathcal{Q} \right\rangle.$$

Thus

$$\begin{aligned} \int_{\overline{\mathcal{Q}} \times \overline{\mathcal{Q}}(\infty,0)} \Theta_n(v, \varphi_f) &= 2 \sum_{\delta \in R_n(\mathcal{Q})} \sum_{\gamma_2 \in \Gamma/\Gamma[\delta^{-1}\mathcal{Q}]} \left\langle \mathcal{Q}(0, \infty), \gamma_2 \delta^{-1} \mathcal{Q} \right\rangle \\ &= 2 \sum_{\delta \in R_n(\mathcal{Q})} \left\langle \overline{\mathcal{Q}}(0, \infty), \overline{\delta^{-1}\mathcal{Q}} \right\rangle \\ &= 2 \left\langle \overline{\mathcal{Q}}(0, \infty), T_n \overline{\mathcal{Q}} \right\rangle \end{aligned}$$

where we use (4.4.13) in the second equality. □

4.4.9 Explicit representatives by Heegner RM-points

We showed that $C_{\mathfrak{a},r} = \overline{\mathcal{Q}}(\alpha_{\mathfrak{a},r}) \times \overline{\mathcal{Q}}(\infty, 0)$. In this section we explain how to compute representatives $\alpha_{\mathfrak{a},r}$ more explicitely. We follow [Pop06, Section. 6] and choose representatives of the narrow class group $\mathrm{Cl}(F)^+$

$$S_F := \{\mathfrak{a}_1, \ldots, \mathfrak{a}_h\} \cup \{\mathfrak{c}\mathfrak{a}_1, \ldots, \mathfrak{c}\mathfrak{a}_h\}$$

such that $\mathfrak{a}_1 := \mathscr{O}$ and for $2 \leq i \leq h$ the $\mathfrak{a}_i$ are prime ideals in $\mathscr{O}$, dividing split primes $q_i := \mathfrak{a}_i \cap \mathbb{Z}$ that are coprime to pN_0. Note that $q_1 = 1$. For $2 \leq i \leq h$ let $r_{q_i,r}$ as in (4.4.5). Since $\gcd(q_i, N_0) = 1$ we can also assume in our choice of roots that $r_{q_i,r} \equiv r \pmod{N_0}$. In order to have a uniform notation we set $r_{q_1,r} = r$. For every i we have $\mathfrak{a}_i = \mathfrak{q}_i$ or $\mathfrak{a} = \mathfrak{q}_i^{\sigma}$, where $\mathfrak{q}_i$ was defined in (4.4.6).

Lemma 4.4.10. *For $\mathfrak{a}_i \in S_F$ we can take*

$$\alpha_{\mathfrak{a}_i,r} = \frac{\sqrt{d_F} - r_{q_i,r}}{q_i N_0} \qquad \text{if} \quad \mathfrak{a}_i = \mathfrak{q}_i,$$
$$\alpha_{\mathfrak{a}_i,r} = \frac{\sqrt{d_F} + r_{q_i,-r}}{q_i N_0} \qquad \text{if} \quad \mathfrak{a}_i = \mathfrak{q}_i^{\sigma}.$$

Proof. First suppose that $\mathfrak{a}_i = \mathfrak{q}_i$. The element $t_{\mathfrak{q}_i} \in F_{q_i}^{\times}$ is given by

$$(t_{\mathfrak{q}_i})_v = (1, 1) \qquad \text{if } v \neq q_i,$$
$$(t_{\mathfrak{q}_i})_{q_i} = (q_i, 1) \qquad \text{if } v = q_i$$

hence by Lemma 4.4.4

$$\begin{aligned}\gamma_r(t_{\mathfrak{q}_i})_v &= 1 \qquad \text{if } v \neq q_i,\\ \gamma_r(t_{\mathfrak{q}_i})_{q_i} &= \begin{pmatrix} \frac{q_i+1}{2} - r\frac{q_i-1}{2\beta_{q_i,r}} & \frac{q_i-1}{\beta_{q_i,r}} \\ -\frac{(q_i-1)N_0}{2\beta_{q_i,r}} & \frac{q_i+1}{2} + r\frac{q_i-1}{2\beta_{q_i,r}} \end{pmatrix}.\end{aligned}$$

Since $\gamma_r(t_{\mathfrak{p}_i})_v = 1$ for $v \neq q_i$ we only have to show that $g_{\mathfrak{q}_i}\gamma_r(t_{\mathfrak{q}_i})_{q_i} \in K_0(p)_{q_i} = \mathrm{GL}_2(\mathbb{Z}_{q_i})$. Note that $\gamma_r(t_{\mathfrak{q}_i}) \in \mathrm{Mat}_2(\mathbb{Z}_{q_i})$ and $\det(\gamma_r(t_{\mathfrak{q}_i})) = q_i$. Moreover, since $\beta_{q_i,r} \equiv r_{q_i,r} \not\equiv 0 \pmod{q_i\mathbb{Z}_{q_i}}$ we have

$$\gamma_r(t_{\mathfrak{q}_i})_{q_i} \equiv \frac{1}{2\beta_{q_i,r}} \begin{pmatrix} r_{q_i,r} + r & -2 \\ N_0 & r_{q_i,r} - r \end{pmatrix} \pmod{q_i\mathbb{Z}_{q_i}}.$$

We multiply by $\begin{pmatrix} 1 & -\frac{r+r_i}{N_0} \\ 0 & 1 \end{pmatrix}$ on the left to make the first line of $\gamma_r(t_{\mathfrak{q}_i})_{q_i}$ divisible by q_i, which we divide by q_i after multiplication by $\begin{pmatrix} q_i^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. Hence for

$$g_{\mathfrak{q}_i,r} \coloneqq \begin{pmatrix} \frac{1}{q_i} & -\frac{r_{q_i,r}+r}{q_iN_0} \\ 0 & 1 \end{pmatrix}$$

we have $g_{\mathfrak{q}_i,r}\gamma_r(t_{\mathfrak{q}_i})_{q_i} \in \mathrm{SL}_2(\mathbb{Z}_{q_i})$ and

$$g_{\mathfrak{q}_i,r}\alpha_r = \frac{\sqrt{d_F} - r_i}{q_iN_0}.$$

The case $\mathfrak{a}_i = \mathfrak{q}_i^\sigma$ is completely analogous, the only difference is that the element $t_{\mathfrak{q}_i^\sigma} \in F_{q_i}^\times$ is given by

$$\begin{aligned}(t_{\mathfrak{q}_i^\sigma})_v &= (1,1) \qquad \text{if } v \neq q_i,\\ (t_{\mathfrak{q}_i^\sigma})_{q_i} &= (1,q_i) \qquad \text{if } v = q_i.\end{aligned}$$

□

Note that these points are roots of the integral quadratic form

$$\frac{q_iN_0}{2}x^2 + r_{q_i,r}x + \frac{r_{q_i,r}^2 - d_F}{2q_iN_0},$$

which has discriminant d_F and of leading coefficient divisible by q_i, *i.e.* they are Heegner RM-points in the sense of [DPV21].

Remark 4.4.6. More precisely we have

$$\overline{\mathcal{Q}}(\psi) = (1 - \psi(\mathfrak{c})) \sum_{i=1}^{h} \psi(\mathfrak{a}_i) \left(\overline{\mathcal{Q}}(\alpha_{\mathfrak{a}_i, r}) + \overline{\mathcal{Q}}(\alpha_{\mathfrak{a}_i, -r}) \right).$$

By the oddness of the character we have $(1 - \psi(\mathfrak{c})) = 2$, otherwise $\overline{\mathcal{Q}}(\psi)$ would be trivial.

4.4.10 Two lattices

Let $\widehat{\mathfrak{d}}^{-1} := \prod'_w \mathfrak{d}_w^{-1}$, where $\mathfrak{d}_w^{-1}$ is the inverse different ideal. Let also $\mathfrak{d}_v^{-1} = \prod_{w|v} \mathfrak{d}_w^{-1}$. Define the following lattices in $\mathbb{A}_F^2$

$$L^{(r)} := \left\{ \begin{pmatrix} x \\ x' \end{pmatrix} \in \widehat{\mathscr{O}}^2 \;\middle|\; x \in \widehat{\mathfrak{d}}^{-1}, \quad w_{\mathfrak{p}}(x') \geq 1, \quad w_{\mathfrak{p}^\sigma}(x') = 0 \right\},$$
$$L^{(-r)} := \left\{ \begin{pmatrix} x \\ x' \end{pmatrix} \in \widehat{\mathscr{O}}^2 \;\middle|\; x \in \widehat{\mathfrak{d}}^{-1}, \quad w_{\mathfrak{p}}(x') = 0, \quad w_{\mathfrak{p}^\sigma}(x') \geq 1 \right\}.$$

Recall the isomorphism

$$\begin{aligned} F_{\mathbb{Q}}^2 &\longrightarrow \mathrm{Mat}_2(\mathbb{Q}) \\ \mathbf{x} = \begin{pmatrix} x \\ x' \end{pmatrix} &\longmapsto [x', SAx], \end{aligned} \tag{4.4.18}$$

defined in (4.4.1), where we identify $F_{\mathbb{Q}} \simeq \mathbb{Q}^2$ via the $\mathbb{Z}$-basis $(\epsilon_r, 1)$ of $\mathscr{O}$. After passing to the adèles and composing with the isomorphism $F_{\mathbb{A}} \simeq \mathbb{A}_F$ we have an isomorphism $\mathbb{A}_F^2 \simeq \mathrm{Mat}_2(\mathbb{A})$.

Lemma 4.4.11. *Under the isomorphism* $\mathbb{A}_F^2 \simeq \mathrm{Mat}_2(\mathbb{A})$ *we have*

$$L^{(r)} \simeq \widehat{\Delta}_0(p).$$

If we replace $(\epsilon_r, 1)$ *with* $(\epsilon_{-r}, 1)$ *then*

$$L^{(-r)} \simeq \widehat{\Delta}_0(p).$$

Proof. We have

$$F_v \oplus F_v = \begin{cases} \mathbb{Q}_v^2 \oplus \mathbb{Q}_v^2 & \text{if } v \text{ split} \\ F \otimes \mathbb{Q}_v \oplus F \otimes \mathbb{Q}_v & \text{otherwise,} \end{cases}$$

where we write the vectors as line vectors instead of column vectors. We decompose $L^{(r)} = \prod_v L_v^{(r)}$

and $\widehat{\mathfrak{d}}^{-1} = \prod_v \mathfrak{d}_v^{-1}$ where

$$L_v^{(r)} = \begin{cases} \mathscr{O}_p \oplus (p\mathbb{Z}_p, \mathbb{Z}_p^\times) & \text{if } v = p \\ \mathscr{O}_v \oplus \mathscr{O}_v & \text{if } v \nmid pd_F \\ \mathfrak{d}_v^{-1} \oplus \mathscr{O}_v & \text{if } v \mid d_F. \end{cases}$$

If $v \mid d_F$, then $(p) = \mathfrak{p}_v^2$ and $\mathfrak{d}_v^{-1} = \mathfrak{p}_v^{-1}$, since $\mathfrak{d}^{-1} = \frac{1}{\sqrt{d_F}}\mathscr{O}$. Similarly

$$L_v^{(-r)} = \begin{cases} \mathscr{O}_p \oplus (\mathbb{Z}_p^\times, p\mathbb{Z}_p) & \text{if } v = p \\ \mathscr{O}_v \oplus \mathscr{O}_v & \text{if } v \nmid pd_F \\ \mathfrak{d}_v^{-1} \oplus \mathscr{O}_v & \text{if } v \mid d_F. \end{cases}$$

Let A_r be the matrix $A = {}^{\mathrm{t}}g_\infty g_\infty$ relative to the basis $(\epsilon_r, 1)$, which is given by

$$A_r = \begin{pmatrix} \frac{r^2+d_F}{2} & r \\ r & 2 \end{pmatrix}.$$

We identity $F \simeq \mathbb{Q}^2$ by $a\epsilon_r + b \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$. We look place by place.

- Suppose that $v = p$. We have $A_r \in \mathrm{GL}_2(\mathbb{Z}_p)$, hence the isomorphism (4.4.18) identifies

$$\begin{gathered} F_{\mathbb{Q}_p}^2 \simeq \mathrm{Mat}_2(\mathbb{Q}_p) \\ \begin{pmatrix} \epsilon_r\mathbb{Z}_p + \mathbb{Z}_p \\ \epsilon_r\mathbb{Z}_p^\times + p\mathbb{Z}_p \end{pmatrix} \simeq \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} = \Delta_0(p)_p \end{gathered}$$

  On the other hand the isomorphism $\varsigma_{p,r} \colon F_{\mathbb{Q}_p} \longrightarrow F_p$ identifies $\epsilon_r\mathbb{Z}_p^\times + p\mathbb{Z}_p$ with $(p\mathbb{Z}_p, \mathbb{Z}_p^\times)$, since we chose r such that $\epsilon_r \in p\mathbb{Z}_p$ and $\epsilon_{-r} \in \mathbb{Z}_p^\times$. Hence we have

$$\begin{gathered} F_{\mathbb{Q}_p}^2 \simeq F_p^2 \\ \begin{pmatrix} \epsilon_r\mathbb{Z}_p + \mathbb{Z}_p \\ \epsilon_r\mathbb{Z}_p^\times + p\mathbb{Z}_p \end{pmatrix} \simeq \mathscr{O}_p \oplus (p\mathbb{Z}_p, \mathbb{Z}_p^\times) = L_p^{(r)}. \end{gathered}$$

  At this place if we replace ϵ_r by ϵ_{-r} we identify $\epsilon_{-r}\mathbb{Z}_p^\times + p\mathbb{Z}_p$ with $(\mathbb{Z}_p^\times, p\mathbb{Z}_p)$, hence $\Delta_0(p)_v \simeq L_v^{(-r)}$.

- Suppose that $v \nmid pd_F$. We have $A_r \in \mathrm{GL}_2(\mathbb{Z}_v)$, hence the isomorphism (4.4.18) identifies

$$\begin{aligned} F^2_{\mathbb{Q}_v} &\simeq \mathrm{Mat}_2(\mathbb{Q}_v) \\ \begin{pmatrix} \epsilon_r \mathbb{Z}_v + \mathbb{Z}_v \\ \epsilon_r \mathbb{Z}_v + \mathbb{Z}_v \end{pmatrix} &\simeq \mathrm{Mat}_2(\mathbb{Z}_v) = \Delta_0(p)_v \end{aligned}$$

On the other hand we have

$$\begin{aligned} F^2_{\mathbb{Q}_v} &\simeq F^2_v \\ \begin{pmatrix} \epsilon_r \mathbb{Z}_v + \mathbb{Z}_v \\ \epsilon_r \mathbb{Z}_v + \mathbb{Z}_v \end{pmatrix} &= \mathscr{O}^2 \otimes \mathbb{Z}_v \simeq \mathscr{O}_v \oplus \mathscr{O}_v = L_v^{(r)}. \end{aligned}$$

- Finally suppose that $v \mid d_F$ is ramified. At this place the isomorphism (4.4.18) is

$$\begin{aligned} F^2_v &\longrightarrow \mathrm{Mat}_2(\mathbb{Q}_v) \\ \begin{pmatrix} c\epsilon_r + d \\ a\epsilon_r + b \end{pmatrix} &\longmapsto \begin{pmatrix} a & -2d - cr \\ b & ds + c\,\mathrm{tr}(\epsilon_r^2) \end{pmatrix}, \end{aligned} \tag{4.4.19}$$

and similarly for ϵ_{-r}. By definition $c\epsilon_r + d \in \mathfrak{d}_v^{-1}$ if and only if $2d + cr$ and $dr + c\,\mathrm{tr}(\epsilon_r^2)$ are both in $\mathbb{Z}_v$. Hence the isomorphism (4.4.19) identifies

$$\begin{aligned} F^2_v &\simeq \mathrm{Mat}_2(\mathbb{Q}_v) \\ L_v^{(r)} = \mathfrak{d}_v^{-1} \oplus \mathscr{O}_v &\simeq \mathrm{Mat}_2(\mathbb{Z}_v) = \Delta_0(p)_v. \end{aligned}$$

□

4.4.11 p-smoothing of Eisenstein series

For a squarefree ideal I we define the Schwartz functions $\varphi^I \in \mathscr{S}(\mathbb{A}_F^2)$ by $\varphi^I_\infty = \varphi_\infty$ as before and

$$\varphi^I_w \begin{pmatrix} x \\ x' \end{pmatrix} := \begin{cases} \mathbf{1}_{\mathfrak{d}_w^{-1}}(x) \left(\mathbf{1}_{\mathscr{O}_w}(x') - \mathbf{1}_{\mathfrak{m}_w}(x')\right) & \text{if} \quad w(I) > 0 \\ \mathbf{1}_{\mathfrak{d}_w^{-1}}(x)\mathbf{1}_{\mathscr{O}_w}(x') & \text{if} \quad w(I) = 0. \end{cases}$$

Let $\phi^I := \mathscr{F}\varphi^I$ be its partial Fourier transform.

Lemma 4.4.12. *We have*

$$\phi^I_w \begin{pmatrix} x \\ x' \end{pmatrix} = \mathrm{vol}(\mathfrak{d}_w^{-1}) \begin{cases} \mathbf{1}_{\mathscr{O}_w}(x) \left(\mathbf{1}_{\mathscr{O}_w}(x') - \mathbf{1}_{\mathfrak{m}_w}(x')\right) & \textit{if} \quad w(I) > 0 \\ \mathbf{1}_{\mathscr{O}_w}(x)\mathbf{1}_{\mathscr{O}_w}(x') & \textit{if} \quad w(I) = 0. \end{cases}$$

Proof. Since $\varphi^I\begin{pmatrix}x\\x'\end{pmatrix} = \varphi_1(x)\varphi_2(x')$ is a product of two Schwartz functions φ_1 and $\varphi_2 \in \mathscr{S}(\mathbb{A}_F)$, we have $\phi^I\begin{pmatrix}x\\x'\end{pmatrix} = \varphi_1^\vee(x)\varphi_2(x')$, where $\varphi_1^\vee(x)$ is the Fourier transform of φ_1. Hence it will follow from the computation

$$\mathbf{1}^\vee_{\mathfrak{m}_w^m\mathfrak{d}_w^{-1}} = q_w^{-m}\operatorname{vol}(\mathfrak{d}_w^{-1})\mathbf{1}_{\mathfrak{m}_w^{-m}\mathscr{O}_w}. \tag{4.4.20}$$

First note that if $\chi\colon K \longrightarrow U(1)$ is a character on a compact group K then

$$\int_K \chi(y)d\mu(y) = \begin{cases}\mu(K) & \text{if } \chi = 1\\ 0 & \text{if } \chi \neq 1.\end{cases}$$

We have

$$\mathbf{1}^\vee_{\mathfrak{m}_w^m\mathfrak{d}_w^{-1}}(x) = \int_{\mathfrak{m}_w^m\mathfrak{d}_w^{-1}} \chi_x(y)d\mu(y),$$

where $\chi_x(y) = \chi(xy)$. The character χ_x is trivial on $\mathfrak{m}_w^m\mathfrak{d}_w^{-1}$ if and only if $x \in \mathfrak{m}_w^{-m}\mathscr{O}_w$ and this proves (4.4.20). □

The Eisenstein series

$$E(\tau_1,\tau_2,\psi) = E(\tau_1,\tau_2,\mathbf{1}_{\widehat{\mathscr{O}}^2},\psi)$$

is a Hilbert modular form of parallel weight 1 for $\mathrm{SL}_2(\mathscr{O})$. Its diagonal restriction $E_1(\tau,\tau,\psi,0)$ vanishes, since it is a weight 2 modular form for $\mathrm{SL}_2(\mathbb{Z})$. For an arbitrary ideal I we define the Eisenstein series

$$E^I(\tau_1,\tau_2,\psi) \coloneqq E(\tau_1,\tau_2,\phi_f^I,\psi),$$

which is of level

$$\Gamma_0(I) = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathscr{O})\,\middle|\, c \in I\right\}.$$

When $I = (p)$ we call it the p-stabilization of $E_1(\tau_1,\tau_2,\psi)$.

Let φ_1^I and φ_2^I be the restriction of φ_f^I to the isotropic lines l_1 and l_2. Since for $w(I) > 0$ we have

$$\varphi_w \begin{pmatrix} x \\ 0 \end{pmatrix} = \mathbf{1}_{\mathfrak{d}_w^{-1}}(x) \left(\mathbf{1}_{\mathscr{O}_w}(0) - \mathbf{1}_{\mathfrak{m}_w}(0)\right) = 0,$$

the function φ_1^I vanishes.

Corollary 4.4.12.1. *We have*

$$E^{(p)}(\tau, \tau, \psi) = \begin{cases} L^{(p)}(\psi, 0) - 4\displaystyle\sum_{n=1}^{\infty} \left\langle \overline{\mathcal{Q}}(0, \infty), T_n \overline{\mathcal{Q}}(\psi) \right\rangle e^{2i\pi n\tau} & \textit{if } p \textit{ is split}, \\ 0 & \textit{if } p \textit{ is inert}. \end{cases}$$

Proof. We can rewrite the Schwartz functions as

$$\varphi_f^{(p)} = \begin{cases} \mathbf{1}_{\widehat{\mathfrak{d}}^{-1} \oplus \widehat{\mathscr{O}}} - \mathbf{1}_{\widehat{\mathfrak{d}}^{-1} \oplus \widehat{\mathfrak{p}}} - \mathbf{1}_{\widehat{\mathfrak{d}}^{-1} \oplus \widehat{\mathfrak{p}^\sigma}} + \mathbf{1}_{\widehat{\mathfrak{d}}^{-1} \oplus p\widehat{\mathscr{O}}} & \text{if } p = \mathfrak{p}\mathfrak{p}^\sigma \\ \mathbf{1}_{\widehat{\mathfrak{d}}^{-1} \oplus \widehat{\mathscr{O}}} - \mathbf{1}_{\widehat{\mathfrak{d}}^{-1} \oplus p\widehat{\mathscr{O}}} & \text{if } p \text{ is inert}, \end{cases}$$

$$\phi_f^{(p)} = d_F^{\frac{1}{2}} \begin{cases} \mathbf{1}_{\widehat{\mathscr{O}}^2} - \mathbf{1}_{\widehat{\mathscr{O}} \oplus \widehat{\mathfrak{p}}} - \mathbf{1}_{\widehat{\mathscr{O}} \oplus \widehat{\mathfrak{p}^\sigma}} + \mathbf{1}_{\widehat{\mathscr{O}} \oplus p\widehat{\mathscr{O}}} & \text{if } p = \mathfrak{p}\mathfrak{p}^\sigma \\ \mathbf{1}_{\widehat{\mathscr{O}}^2} - \mathbf{1}_{\widehat{\mathscr{O}} \oplus p\widehat{\mathscr{O}}} & \text{if } p \text{ is inert}. \end{cases}$$

First note that we have

$$\omega_l \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \mathbf{1}_{\widehat{\mathscr{O}}^2} = p^{\frac{1}{2}} \mathbf{1}_{\widehat{\mathscr{O}} \oplus p\widehat{\mathscr{O}}}.$$

By applying the transformation in Proposition 4.1.2 we find that

$$E(\tau_1, \tau_2, \mathbf{1}_{\widehat{\mathscr{O}} \oplus p\widehat{\mathscr{O}}}, \psi) = p^2 E\left(p\tau_1, p\tau_2, \mathbf{1}_{\widehat{\mathscr{O}}^2}, \psi\right).$$

Hence if p is inert, we have

$$E_1^{(p)}(\tau, \tau, \psi) = d_F^{\frac{1}{2}} \left(E(\tau, \tau, \psi) - p^2 E\left(p\tau, p\tau, \psi\right)\right) = 0,$$

since $E(\tau, \tau)$ is zero.

Suppose that p is split. We have $\mathbf{1}_{L^{(r)}} = \mathbf{1}_{\widehat{\mathfrak{d}}^{-1} \oplus \widehat{\mathfrak{p}}} - \mathbf{1}_{\widehat{\mathfrak{d}}^{-1} \oplus p\widehat{\mathscr{O}}}$ and $\mathbf{1}_{L^{(-r)}} = \mathbf{1}_{\widehat{\mathfrak{d}}^{-1} \oplus \widehat{\mathfrak{p}^\sigma}} - \mathbf{1}_{\widehat{\mathfrak{d}}^{-1} \oplus p\widehat{\mathscr{O}}}$, hence

$$\varphi_f^{(p)} + \mathbf{1}_{L^{(r)}} + \mathbf{1}_{L^{(-r)}} = \mathbf{1}_{\widehat{\mathfrak{d}}^{-1}\oplus\widehat{\mathscr{O}}} - \mathbf{1}_{\widehat{\mathfrak{d}}^{-1}\oplus p\widehat{\mathscr{O}}},$$
$$\phi_f^{(p)} + \mathscr{F}\mathbf{1}_{L^{(r)}} + \mathscr{F}\mathbf{1}_{L^{(-r)}} = d_F^{\frac{1}{2}}\left(\mathbf{1}_{\widehat{\mathscr{O}}^2} - \mathbf{1}_{\widehat{\mathscr{O}}\oplus p\widehat{\mathscr{O}}}\right).$$

This implies

$$E^{(p)}(\tau_1,\tau_2,\psi) = -E(\tau_1,\tau_2,\mathscr{F}\mathbf{1}_{L^{(r)}},\psi) - E(\tau_1,\tau_2,\mathscr{F}\mathbf{1}_{L^{(-r)}},\psi),$$

which also means that for the respective constant terms we have

$$c_0(\varphi^{(p)}) = -c_0(\mathbf{1}_{L^{(r)}}) - c_0(\mathbf{1}_{L^{(-r)}}).$$

By Theorem 4.3.12 and the correspondence $L^{(r)} \simeq \widehat{\Delta}_0(p)$ in Lemma 4.4.11 we have

$$E(\tau,\tau,\mathscr{F}\mathbf{1}_{L^{(r)}},\psi) = c_0(\mathbf{1}_{L^{(r)}}) + 2\sum_{n=1}^{\infty}\left(\int_{C_r\otimes\psi}\Theta_n(v,\mathbf{1}_{\widehat{\Delta}_0(p)})\right)e^{2i\pi n\tau}$$

and similarly for $-r$. Putting the two together we get

$$E^{(p)}(\tau,\tau,\psi) = c_0(\varphi^{(p)}) - 2\sum_{n=1}^{\infty}\left(\int_{C_r\otimes\psi+C_{-r}\otimes\psi}\Theta_n(v,\mathbf{1}_{\widehat{\Delta}_0(p)})\right)e^{2i\pi n\tau}.$$

Since $C_r\otimes\psi + C_{-r}\otimes\psi = \overline{\mathcal{Q}}(\psi)\times\overline{\mathcal{Q}}(\infty,0)$ it follows from Proposition 4.4.9 that

$$E^{(p)}(\tau,\tau,\psi) = c_0(\varphi^{(p)}) - 4\sum_{n=1}^{\infty}\langle\overline{\mathcal{Q}}(0,\infty), T_n\overline{\mathcal{Q}}(\psi)\rangle e^{2i\pi n\tau}.$$

It remains to compute $c_0(\varphi^{(p)}) = \zeta_f(\varphi_1^{(p)},\psi^{-1},0) + \zeta_f(\varphi_2^{(p)},\psi,0)$. The Schwartz function $\varphi_f^{(p)}$ vanishes on l_1 hence the first singular term of $c_0(\varphi^{(p)})$ is zero and we have

$$c_0(\varphi^{(p)}) = \zeta_f(\varphi_2^{(p)},\psi,0),$$

where

$$\varphi_{2,w}^{(p)}(x') = \begin{cases}\mathbf{1}_{\mathscr{O}_w^\times}(x') & \text{if } w(p)>0\\ \mathbf{1}_{\mathscr{O}_w}(x') & \text{if } w(p)=0.\end{cases}$$

By the computation in (4.1.6) this shows that

$$\zeta_w(\varphi_2^{(p)}, \psi, s) = \begin{cases} 1 & \text{if } w(p) > 0, \\ L_w(\psi, s) & \text{if } w(p) = 0. \end{cases}$$

hence $\zeta_f(\varphi_2^{(p)}, \psi, 0) = L^{(p)}(\psi, 0)$. □

Remark 4.4.7. Our formula in Corollary 4.4.12.1 differs by a factor of 2 from [DPV21, theorem. A]: the factor 4 that we obtain in front of the positive Fourier coefficients is a factor 2 in *loc. cit.* We already mentionned that this is due to the absence of the factor κ in *loc.cit.* but let us make this more precise. The difference comes from the different definitions of intersection numbers of geodesic. Let $\overline{\mathcal{Q}}$ be the (compact) image of the geodesic $\mathcal{Q}$ in $Y_0(p)$. The subgroup of Γ stabilizing $\mathcal{Q}$ is

$$\Gamma[\mathcal{Q}] = \{\pm 1\} \times \gamma_{\mathcal{Q}}{}^{\mathbb{Z}}$$

for some $\gamma_{\mathcal{Q}} \in \Gamma$. For some $\tau \in \mathcal{Q}$ let $[\tau, \gamma_{\mathcal{Q}}\tau)$ be the (half-open) geodesic segment in $\mathbb{H}$ between τ and $\gamma_{\mathcal{Q}}\tau$. In our case - see (4.4.15) - the definition of intersection number between $\overline{\mathcal{Q}}(0, \infty)$ and $\overline{\mathcal{Q}}$ that we use is

$$\left\langle \overline{\mathcal{Q}}(0, \infty), \overline{\mathcal{Q}} \right\rangle = \sum_{\gamma \in P\Gamma} \left\langle [\gamma\tau, \gamma\gamma_{\mathcal{Q}}\tau), \mathcal{Q} \right\rangle, \tag{4.4.21}$$

where $P\Gamma$ is the image of Γ in $\mathrm{PSL}_2(\mathbb{Z})$. On the other hand, in [DPV21] the intersection numbers are defined by

$$\left\langle \overline{\mathcal{Q}}(0, \infty), \overline{\mathcal{Q}} \right\rangle = \sum_{\gamma \in \Gamma} \left\langle \mathcal{Q}(0, \infty), [\gamma\tau, \gamma\gamma_{\mathcal{Q}}\tau) \right\rangle$$

which is twice the number in (4.4.21).

Chapter 5

Denominators of Eisenstein classes

In [BCG20; BCG21] Bergeron-Charollois-Garcia use the Mathai-Quillen form to construct Eisenstein classes on the locally symmetric space associated to $\mathrm{SL}_N(K)$ for some imaginary quadratic field K. We relate this class to Harder's Eisenstein cohomology, which allows us to give an upper bound for its denominator in terms of the special value of an L-function; see Theorem 5.3.5 (Theorem C in the introduction).

5.1 Eisenstein series, L-functions and Sczech's cocycle

Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic imaginary field with no non-trivial units, let d_K be the fundamental discriminant of K. Let $\mathscr{O} = \mathbb{Z} + \omega\mathbb{Z}$ be its ring of integers, which we view as a lattice in $\mathbb{C}$ after fixing some embedding $K \subset \mathbb{C}$. More generally consider a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ of K where ω_1 and ω_2 are complex numbers with $\omega_1/\omega_2 \in \mathbb{H}$ and define

$$D(L) := I(\omega_1 \overline{\omega}_2)$$

where $I(z) := z - \bar{z}$. We define the order of L

$$\mathscr{O}(L) := \{ \lambda \in \mathbb{C} | \lambda L = L \} .$$

It is an order of $\mathscr{O}$, and the lattice L is homothetic to an ideal in $\mathscr{O}(L)$. In particular, if $L = \mathfrak{a}$ is an ideal then $\mathscr{O}(\mathfrak{a}) = \mathscr{O}$.

Consider the the elliptic curve

$$E_L \colon y^2 = x^3 - g_2(L)x - g_3(L)$$

where

$$g_2(L) := 60 \sum_{\omega \in L}' \frac{1}{\omega^4}, \qquad g_3(L) := 140 \sum_{\omega \in L}' \frac{1}{\omega^6}.$$

We have the Weierstrass isomorphism

$$\begin{aligned} \mathbb{C}/L &\longrightarrow E_L(\mathbb{C}) \\ z &\longmapsto (\wp(z), \wp(z')), \end{aligned} \tag{5.1.1}$$

where the Weierstrass $\wp$-function is defined for $z \in \mathbb{C} - L$

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in L}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Conversely, suppose we start with the elliptic curve

$$E \colon y^2 = 4x^3 - g_2 x - g_3, \qquad g_2, g_3 \in \overline{\mathbb{Q}}$$

that has complex multiplication by $\mathscr{O}$. The elliptic curve is defined over the field

$$K_E := \mathbb{Q}(g_2, g_3),$$

and let $\mathscr{O}_E$ be the ring of integers of K_E. For $\alpha \in \mathscr{O}_E$ we can make the change of variable $u = \alpha^2$ and $v = \alpha^3 y$ so that $E \colon u^2 = v^3 - \tilde{g}_2 v - \tilde{g}_3$ where $\tilde{g}_2 = \alpha^4 g_2$ and $\tilde{g}_3 = \alpha^6 g_3$. Hence, without loss of generality we can also take g_2 and g_3 in $\mathscr{O}_E$.

The inverse of the isomorphism (5.1.1) is given by

$$\begin{aligned} E(\mathbb{C}) &\longmapsto \mathbb{C}/L_E \\ Q &\longmapsto \int_P^Q \frac{dx}{2y} \mod L_E \end{aligned}$$

for some point $P \in E(\mathbb{C})$ and

$$L_E := \left\{ \int_\gamma \frac{dx}{2y} \middle| \gamma \in H_1(E(\mathbb{C});\mathbb{Z}) \right\}.$$

Moreover, we have $g_k = g_k(L_E)$ for $k = 2$ and $k = 3$. Since E has complex multiplication by $\mathscr{O}$, we have $\mathscr{O}(L_E) = \mathscr{O}$ and the lattice L_E is homothetic to some ideal $\mathfrak{a} \subset \mathscr{O}$. Let $\Omega \in \mathbb{C}$ be such that

$$L_E = \Omega\mathfrak{a}.$$

If K has class number 1 then we can take $\mathfrak{a} = \mathscr{O}$ and we get the following.

Proposition 5.1.1. *Suppose that K has class number 1. There is a complex period Ω such that $g_2(L)$ and $g_3(L)$ are in $\mathscr{O}_E$ for $L = \Omega\mathscr{O}$.*

Kronecker-Eisenstein series Consider the character

$$\theta(z) := \exp\left(2i\pi \frac{I(z)}{D(L)}\right).$$

For a non-negative integer k and $\lambda \in \mathbb{C}$ we define the Kronecker-Eisenstein series

$$G(s,k,p,q,L) := \sum_{\omega \in L}{}' \theta(w\bar{p}) \frac{\overline{q+\omega}^k}{|q+\omega|^{2s+k}},$$

which converges for $\mathrm{Re}(s) > 1$ and the $'$ means that we remove $q = -\omega$ from the summation if $q \in L$. This is the series considered by Weil in [Wei76, section VIII]. The function admits a meromorphic continuation to the whole plane with only possible poles at $s = 0$ (if $k = 0$ and $q \in L$) and at $s = \frac{1}{2}$ (if $k = 0$ and $p \in L$); see [Wei76, section VIII, p. 80]. Moreover it satisfies the functional equation

$$\mathcal{E}(s,k,p,q,L) = \theta(p\bar{q})\mathcal{E}(1-s,k,p,q,L) \tag{5.1.2}$$

where

$$\mathcal{E}(s,k,p,q,L) := \left(\frac{2i\pi}{D(L)}\right)^{-s}\Gamma\left(s+\frac{k}{2}\right)G(s,k,p,q,L).$$

We define for $k>0$

$$\begin{aligned} G_k(z,L) &:= G\left(\frac{k}{2},k,0,z,L\right) \\ &= \sum_{\omega\in L}{}' \frac{1}{(z+\omega)^k|z+k|^\lambda}\Bigg|_{\lambda=0} \end{aligned}$$

and

$$G(z,L) := \frac{2i\pi}{D(L)}G(0,2,0,z).$$

For $z=0$ we set

$$G(L) := G(0,L), \qquad G_k(L) := G_k(0,L).$$

We have

$$\begin{aligned} G_k(\alpha z,\alpha L) &= \alpha^{-k}G_k(z,L) \\ G(\alpha z,\alpha L) &= \frac{\bar\alpha}{\alpha}G(z,L). \end{aligned} \tag{5.1.3}$$

Sczech cocycle For a and c in $\mathscr{O}$ and c nonzero we define the Dedekind sum

$$D(a,c,L) := \frac{1}{c}\sum_{r\in L/cL} G_1\left(\frac{ar}{c},L\right)G_1\left(\frac{r}{c},L\right).$$

In [Scz84], Sczech shows that the map $\Phi_L\colon \mathrm{SL}_2(\mathscr{O}) \longrightarrow \mathbb{C}$ defined by

$$\Phi_L\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{cases} I\left(\frac{a+d}{c}\right)G_2(L) - D(a,c,L) & \text{if} \quad c\neq 0, \\ I\left(\frac{b}{d}\right)G_2(L) & \text{if} \quad c=0 \end{cases}$$

is a cocycle.

Hecke characters and L-functions Let $\psi\colon \mathcal{I}(\mathfrak{f}) \longrightarrow \mathbb{C}^\times$ be an algebraic Hecke character of infinity type $z^a\bar z^b$. Recall that this is a character

$$\psi\colon \mathcal{I}(\mathfrak{f}) \longrightarrow F_\psi \subset \mathbb{C}^\times$$

with values in a finite extension F_ψ of K such that at principal ideals in $\mathcal{P}_{\mathfrak{f}}(K)^+$ we have

$$\psi((\alpha)) = \alpha^{-a}\bar{\alpha}^{-b}.$$

To such a character we can attach an L-function

$$L(\psi, s) := \sum_{0\neq\mathfrak{a}\subseteq\mathscr{O}} \frac{\psi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s},$$

which converges for $\mathrm{Re}(s) > 1$, admits a meromorphic continuation to the whole plane and a functional equation. We define the algebraic L-function

$$L^{\mathrm{alg}}(\psi, s) := \Omega^{b-a}\left(\frac{2\pi}{\sqrt{d_K}}\right)^{-b}\Gamma(a)L(\psi, s),$$

where d_K is the fundamental discriminant and Ω is as in Proposition 5.1.1 such that $g_2(L), g_3(L)$ are integral with $L = \Omega\mathscr{O}$. Note that when K has class number 1 and ψ is of infinity type z^k we have

$$L(\psi, s) = \frac{1}{|\mathscr{O}^\times|}\sum_{\alpha\in\mathscr{O}-\{0\}} \frac{\alpha^{-k}}{(\alpha\bar{\alpha})^s}$$

and at $s = 0$

$$L(\psi, 0) = \frac{1}{|\mathscr{O}^\times|}G_k(\mathscr{O}).$$

The normalization is chosen such that we have the following.

Proposition 5.1.2. *We have $L^{\mathrm{alg}}(\psi, 0) \in \overline{\mathbb{Q}}$. When K has class number 1 and ψ is of infinity type z^2 then $|\mathscr{O}^\times|\sqrt{d_K}L^{\mathrm{alg}}(\psi, 0)$ is an algebraic integer.*

Proof. Let us prove the second statement, when the class number is 1 and the character is of infinity type z^2. In that case $L(\psi, 0) = \frac{1}{|\mathscr{O}^\times|}G_2(\mathscr{O})$. For $L = \Omega\mathscr{O}$, by (5.1.3) we have

$$G_2(L) = |\mathscr{O}^\times|\Omega^{-2}G_2(\mathscr{O}) = |\mathscr{O}^\times|\Omega^{-2}L(\psi, 0) = |\mathscr{O}^\times|L^{\mathrm{alg}}(\psi, 0),$$

and in Proposition 5.3.2 we will show that $\sqrt{d_K}G_2(L)$ is integral. $\square$

5.2 Eisenstein cohomology

Let $\mathbb{H}_3$ be the hyperbolic 3-space

$$\mathbb{H}_3 \coloneqq \{u = z + jv \mid z \in \mathbb{C}, v \in \mathbb{R}_{>0}\}$$

where $ij = -ji$ and $i^2 = j^2 = -1$. For $u = z + jv$ let $\overline{u} = \overline{z} - jv$ and $|u| = u\overline{u} = |z|^2 + v^2$. The group $\mathrm{SL}_2(\mathbb{C})$ acts transitively on $\mathbb{H}^3$ by

$$u \longmapsto (au+b)(cu+d)^{-1} = \frac{(au+b)(\overline{cu+d})}{|cz+d|^2+v^2},$$

and the stabilizer of j is $\mathrm{SU}(2)$. Hence the symmetric space $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$ is isomorphic to $\mathbb{H}^3$. For a fractional ideal $\mathfrak{b}$ of K let

$$\Gamma(\mathfrak{b}) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K) \,\middle|\, a, d \in \mathscr{O}, b \in \mathfrak{b}^{-1}, c \in \mathfrak{b} \right\}$$

be the subgroup of $\mathrm{SL}_2(K)$ preserving $\mathscr{O} \oplus \mathfrak{b}$. From now on we set $\Gamma \coloneqq \Gamma(\mathscr{O}) = \mathrm{SL}_2(\mathscr{O})$ and

$$Y_\Gamma \coloneqq \Gamma \backslash \mathbb{H}_3.$$

This space is a non-compact 3-dimensional orbifold and can be compactified in several ways, one of them is the Borel-Serre compactification.

5.2.1 Borel-Serre compactification

We describe the Borel-Serre compactification of Y_Γ; see [BJ06] or[JM02] for more on compactifications of locally symmetric spaces. Define the space

$$\mathbb{H}_3^* \coloneqq \mathbb{H}_3 \cup \bigsqcup_{r \in \mathbb{P}^1(K)} \mathcal{H}_r$$

where $\mathcal{H}_r = \mathbb{P}^1(\mathbb{C}) - \{r\} \simeq \mathbb{C}$ is a horocycle at the cusp r. If $r = (m : n)$ then we can view this space as adding a copy of $\mathbb{C}$ at the cusp $\frac{m}{n}$. The topology on $\mathbb{H}_3^*$ is defined as follows: let $\mathcal{H}_\infty$ be the horocycle at ∞ corresponding to $r = (1 : 0)$. A sequence $u_n = z_n + jv_n$ converges to $z_0 \in \mathcal{H}_\infty$ if $\lim_{n\to\infty} v_n = \infty$ and $\lim_{n\to\infty} z_n = z_0$. If γ maps ∞ to r then u_n converges to $z_0 \in \mathcal{H}_r$ if $\gamma^{-1}u_n$ converges to $\gamma^{-1}z_0 \in \mathcal{H}_\infty$. The action of Γ extends to $\mathbb{H}_3^*$ by sending $z \in \mathcal{H}_r$ to $\gamma z \in \mathcal{H}_{\gamma r}$, where

$\mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x:y) = (ax+by : cx+dy).$$

We define $X_\Gamma \coloneqq \Gamma\backslash\mathbb{H}_3^*$. Let

$$\mathcal{C}_\Gamma \coloneqq \Gamma\backslash\mathbb{P}^1(K)$$

be the set of equivalence class of cusps of Y_Γ. We can represent cusps by fractional ideals in K since we have a bijection [Fre90, Lemma. 3.5]

$$\begin{aligned} \mathcal{C}_\Gamma &\longrightarrow \mathrm{Cl}(K) \\ \Gamma(m:n) &\longmapsto \mathfrak{a} \coloneqq m\mathscr{O} + n\mathscr{O} \end{aligned}$$

where $\mathrm{Cl}(K)$ is the class group. We have a bijection

$$\begin{aligned} \Gamma\backslash \bigsqcup_{r\in\mathbb{P}^1(K)} \mathcal{H}_r &\longrightarrow \bigsqcup_{\mathfrak{a}\in\mathcal{C}_\Gamma} \Gamma_\mathfrak{a}\backslash\mathcal{H}_\mathfrak{a} \\ (x:y) &\longmapsto \gamma(x:y) \end{aligned}$$

where $(x:y)\in\mathcal{H}_r$, the element γ maps $(1:0)$ to some $\mathfrak{a}\in\mathcal{C}_\Gamma$ and $\Gamma_\mathfrak{a}$ is the stabilizer of $\mathfrak{a}$ in Γ. For every $\mathfrak{a}\in\mathcal{I}(K)$ let $A_\mathfrak{a}\in\mathrm{SL}_2(K)$ be such that $A_\mathfrak{a}\mathfrak{a}=\infty$ i.e. such that

$$(m_\mathfrak{a} : n_\mathfrak{a}) = A_\mathfrak{a}^{-1}(1:0)$$

where $\mathfrak{a}\coloneqq m_\mathfrak{a}\mathscr{O} + n_\mathfrak{a}\mathscr{O}$. We can find $x,y\in\mathfrak{a}^{-1}$ such that $m_\mathfrak{a}y - n_\mathfrak{a}x = 1$ and take

$$A_\mathfrak{a}^{-1} = \begin{pmatrix} m_\mathfrak{a} & x \\ n_\mathfrak{a} & y \end{pmatrix} \in \begin{pmatrix} \mathfrak{a} & \mathfrak{a}^{-1} \\ \mathfrak{a} & \mathfrak{a}^{-1} \end{pmatrix}.$$

It follows that

$$A_\mathfrak{a}\Gamma A_\mathfrak{a}^{-1} = \Gamma(\mathfrak{a}^2).$$

Let

$$\Gamma(\mathfrak{b})_\infty \coloneqq \left\{ \begin{pmatrix} \pm1 & z \\ 0 & \pm1 \end{pmatrix} \in \mathrm{SL}_2(\mathscr{O}) \,\middle|\, z\in\mathfrak{b}^{-1} \right\},$$

the stabilizer of the cusp ∞ in $\Gamma(\mathfrak{b})$. Then

$$A_{\mathfrak{a}}\Gamma_{\mathfrak{a}}A_{\mathfrak{a}}^{-1} = \left(A_{\mathfrak{a}}\Gamma A_{\mathfrak{a}}^{-1}\right)_{\infty} = \Gamma(\mathfrak{a}^2)_{\infty}.$$

We have bijections

$$\begin{aligned} \Gamma(\mathfrak{b})_{\infty}\backslash\mathcal{H}_{\infty} &\longrightarrow \mathbb{C}/\mathfrak{b} \\ (m:n) &\longmapsto \frac{m}{n} \end{aligned}$$

and

$$\begin{aligned} \Gamma_{\mathfrak{a}}\backslash\mathcal{H}_{\mathfrak{a}} &\longrightarrow \Gamma(\mathfrak{a}^2)_{\infty}\backslash\mathcal{H}_{\infty} \\ \lambda &\longmapsto A_{\mathfrak{a}}\lambda. \end{aligned}$$

Hence, by composing the two previous maps we get a map

$$\phi_{\mathfrak{a}} \colon \Gamma_{\mathfrak{a}}\backslash\mathcal{H}_{\mathfrak{a}} \longrightarrow \mathbb{C}/\mathfrak{a}^2$$

and

$$X_{\Gamma} = Y_{\Gamma} \cup \bigsqcup_{[\mathfrak{a}]\in \mathrm{Cl}(K)} \mathbb{C}/\mathfrak{a}^2.$$

The boundary of the Borel-Serre compactification is

$$\partial X_{\Gamma} = \bigsqcup_{[\mathfrak{a}]\in \mathrm{Cl}(K)} \mathbb{C}/\mathfrak{a}^2.$$

5.2.2 Equivariant homology and cohomology

Since Y_{Γ} is not a manifold but rather an orbifold we need to work with equivariant (co)homology.

Cohomology Following [Ste89, Section. 2] we define a k-form ω on $\mathbb{H}_3^*$ to be a k-form ω_0 on $\mathbb{H}_3$ and a family k-forms $\omega^{(r)}$ on $\mathcal{H}_r$ such that $\lim_{z\to r}\omega_0 = \omega^{(r)}$. We denote by $\Omega^k(\mathbb{H}_3^*)$ the space of such forms. Let

$$\Omega^k(\mathbb{H}_3^*;\mathbb{C}) \coloneqq \Omega^k(\mathbb{H}_3^*)\otimes_{\mathbb{R}}\mathbb{C}$$

be the differential forms valued in $\mathbb{C}$. Let $\Omega^k(\mathbb{H}_3^*;\mathbb{C})^{\Gamma}$ be the complex of Γ-invariant forms, consisting of forms that satisfy $\gamma^*\omega_0 = \omega_0$ and $\gamma^*\omega^{(r)} = \omega^{(\gamma^{-1}r)}$. Let $H^k(X_{\Gamma};\mathbb{C}) = H^k(\Omega(\mathbb{H}_3^*;\mathbb{C})^{\Gamma})$ be the cohomology of this complex. Similarly we have

$$H^k(\Gamma_{\mathfrak{a}}\backslash\mathcal{H}_{\mathfrak{a}};\mathbb{C}) = H^k(\Omega^\bullet(\mathcal{H}_{\mathfrak{a}};\mathbb{C})^{\Gamma_{\mathfrak{a}}})$$

and

$$H^k(\partial X_\Gamma;\mathbb{C}) = \bigoplus_{[\mathfrak{a}]\in \mathrm{Cl}(K)} H^k(\Gamma_{\mathfrak{a}}\backslash\mathcal{H}_{\mathfrak{a}};\mathbb{C}).$$

A class $[\omega] \in H^k(X_\Gamma;\mathbb{C})$ is represented by a closed Γ-invariant form ω_0 on $\mathbb{H}^3$ and a family of $\Gamma_{\mathfrak{a}}$-invariant forms $(\omega^{(\mathfrak{a})})_{[\mathfrak{a}]\in\mathrm{Cl}(K)}$ on $\mathcal{H}_{\mathfrak{a}}$. The pullback by the inclusion $\partial X_\Gamma \hookrightarrow X_\Gamma$ gives a restriction map

$$\begin{aligned} \mathrm{res}\colon H^k(X_\Gamma;\mathbb{C}) &\longrightarrow H^k(\partial X_\Gamma;\mathbb{C}) \\ [\omega] &\longmapsto ([\omega^{(\mathfrak{a})}])_{[\mathfrak{a}]\in\mathrm{Cl}(K)}. \end{aligned}$$

Since the compactification X_Γ is homotopy equivalent to Y_Γ, we have $H^k(X_\Gamma;\mathbb{C}) \simeq H^k(Y_\Gamma;\mathbb{C})$ and we get a map

$$\mathrm{res}\colon H^k(Y_\Gamma;\mathbb{C}) \longrightarrow H^k(\partial X_\Gamma;\mathbb{C}).$$

Relative cohomology Let $\Omega^k(\mathbb{H}_3^*,\partial\mathbb{H}_3^*;\mathbb{C})^\Gamma := \Omega^k(\mathbb{H}_3^*;\mathbb{C})^\Gamma \oplus \Omega^{k-1}(\partial\mathbb{H}_3^*;\mathbb{C})^\Gamma$ be the complex with the coboundary operator

$$\begin{aligned} \delta\colon \Omega^k(\mathbb{H}_3^*,\partial\mathbb{H}_3^*;\mathbb{C})^\Gamma &\longrightarrow \Omega^{k+1}(\mathbb{H}_3^*,\partial\mathbb{H}_3^*;\mathbb{C})^\Gamma \\ (\omega,\theta) &\longmapsto (d\omega, \iota^*\omega - d\theta). \end{aligned}$$

Let $H^k(X_\Gamma,\partial X_\Gamma;\mathbb{C})$ the cohomology associated to this complex, *the cohomology of X_Γ relative to ∂X_Γ*. We have an exact sequence

$$0 \longrightarrow \Omega^{k-1}(\partial\mathbb{H}_3^*;\mathbb{C})^\Gamma \xhookrightarrow{\ \alpha\ } \Omega^k(\mathbb{H}_3^*,\partial\mathbb{H}_3^*;\mathbb{C})^\Gamma \xrightarrow{\ \beta\ } \Omega^k(\mathbb{H}_3^*;\mathbb{C})^\Gamma \longrightarrow 0$$

where the first map is given by $\alpha(\theta) = (0,\theta)$ and the second by $\beta(\omega,\theta) = \omega$. This induces a long exact sequence in cohomology

$$H^{k-1}(\partial X_\Gamma;\mathbb{C}) \xrightarrow{\alpha^*} H^k(X_\Gamma,\partial X_\Gamma;\mathbb{C}) \xrightarrow{\beta^*} H^k(X_\Gamma;\mathbb{C}) \xrightarrow{\mathrm{res}}$$
$$H^k(\partial X_\Gamma;\mathbb{C}) \xrightarrow{\alpha^*} H^{k+1}(X_\Gamma,\partial X_\Gamma;\mathbb{C}) \xrightarrow{\beta^*} H^{k+1}(X_\Gamma;\mathbb{C})$$

and the boundary map res is the restriction to the boundary.

Homology For an abelian group A let $C_n(\mathbb{H}_3; A)$ be the $\mathbb{Z}$-module of singular n-chains valued in A. The action of Γ on $\mathbb{H}_3$ endows $C_n(\mathbb{H}_3; A)$ with a Γ-module structure and we define the complex of coinvariant chains

$$C_n(\mathbb{H}_3; A)_\Gamma \coloneqq C_n(\mathbb{H}_3; A)/\langle \sigma - \gamma\sigma \rangle$$

where we quotient by the submodule generated by all $\sigma - \gamma\sigma$ with $\sigma \in C_n(\mathbb{H}_3; A)$. Let $H_n(Y_\Gamma; A)$ be the homology of this complex. Similalry we define $H_n(X_\Gamma; A)$.

Let $u_0 \in \mathbb{H}_3$ be some basepoint and $[u_0, \gamma u_0]$ be a path segment joining u_0 and γu_0 for $\gamma \in \Gamma$. The boundary of $[u_0, \gamma u_0]$ is $\gamma u_0 - u_0$ hence it represents a class in $H_1(Y_\Gamma; \mathbb{C})$.

Proposition 5.2.1. *The map $\Gamma \longrightarrow H_1(Y_\Gamma; \mathbb{Z})$ sending γ to $[u_0, \gamma u_0]$ is a surjective morphism and independant of u_0.*

Proof. If Y_Γ were a manifold (for example if Γ were some congruence subgroup of $\mathrm{SL}_2(\mathscr{O})$) we could work with singular homology. Then the map would be the Hurewicz homomorphism, which is surjective since Y_Γ is path connected; see [Hat01, Theorem. 2A.1]. In the case of equivariant homology it works almost in the same way and we follow the proof of [Hat01].

We write $[a, a'] \sim [b, b']$ for two homologous paths joining points a, a', b and b' in $\mathbb{H}_3$. We have the following relations:

1. $[a, a] \sim 0$,
2. $[a, b] + [b, c] \sim [a, c]$
3. $[a, b] \sim -[b, a]$
4. $[a, \gamma a] \sim [b, \gamma b]$,
5. $[a, b] \sim [\gamma a, \gamma b]$.

The first two ones are clear since $[a, a]$ is the boundary of the constant 2-simplex $C = \{a\}$ and $[a, b] - [a, c] + [b, c]$ is the boundary of the simplex joining a, b and c. The third one follows from 1. and 2. For 4. consider the two simplices C_1 and C_2 as in Figure 5.1. The boundaries are given by

$$\partial C_1 = [\gamma a, \gamma b] - [a, \gamma b] + [a, \gamma a]$$
$$\partial C_2 = [b, \gamma b] - [a, \gamma b] + [a, b]$$

so that

$$\partial(C_1 - C_2) = [a, \gamma a] - [b; \gamma b] + [\gamma a, \gamma b] - [a, b] = [a, \gamma a] - [b, \gamma b]$$

where $[\gamma a, \gamma b] - [a, b] = 0$ since we are in the complex of coinvariants. For 5. consider the simplices

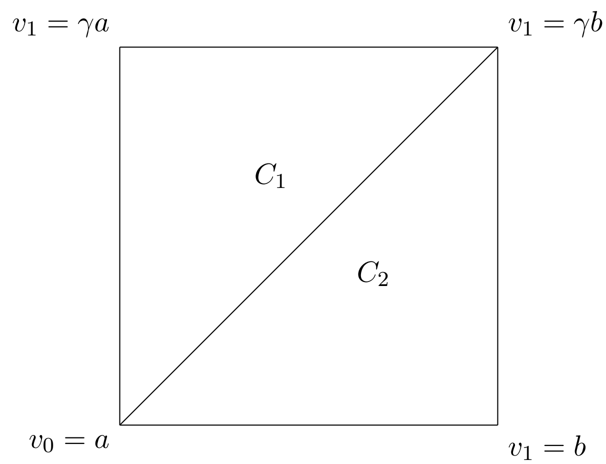


Figure 5.1: $[a, \gamma a] \sim [b, \gamma b]$

C_1 and C_2 as in Figure 5.2. The boundaries are given by

$$\partial C_1 = [b, \gamma a] - [\gamma b, \gamma a] + [\gamma b, b]$$
$$\partial C_2 = [b, \gamma a] - [a, \gamma a] + [a, b]$$

so that

$$\partial(C_1 - C_2) = [\gamma a, \gamma b] - [b, \gamma b] + [a, \gamma a] - [a, b] = [\gamma a, \gamma b] - [a, b]$$

Let us now prove the statement of the proposition. The fact that it is independant of the basepoint follows from 4., and the fact that it is a homorphism follows from 2. and 4. since

$$[u_0, \gamma_1 u_0] + [u_0, \gamma_2 u_0] \sim [\gamma_2 u_0, \gamma_1 \gamma_2 u_0] + [u_0, \gamma_2 u_0] \sim [u_0, \gamma_1 \gamma_2 u_0].$$

For surjectivity suppose we have a class represented by a cycle

$$\sigma = \sum_{i=1}^{m} n_i [a_i, b_i] \in C_1(\mathbb{H}_3)_\Gamma.$$

After relabeling the paths we can suppose that $n_i = \pm 1$ and using 3. we can suppose that $n_i = 1$.

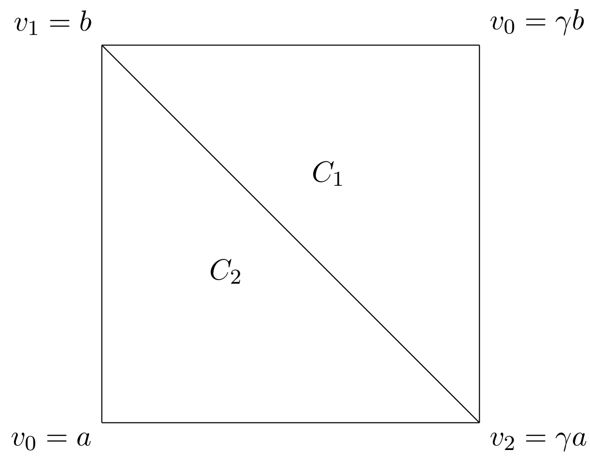


Figure 5.2: $[a, b] \sim [\gamma a, \gamma b]$

Since the boundary is

$$\partial\sigma = \sum_i (b_i - a_i) = 0$$

we necessarily have that every b_j is equal to $\gamma_{ij} a_i$ for a unique a_i and some $\gamma_{ij} \in \Gamma$. We can see it as a permutation on the set $\{1, \ldots, m\}$, where we send i to j if a_i is Γ-equivalent to b_j.

First suppose that the corresponding permutation is the identity $i.e$ $b_i = \gamma_i a_i$ for every i . Then using 4. and the fact that the map is a morphism we get

$$\sigma = \sum_{i=1}^{m} [a_i, \gamma_i a_i] \sim \sum_{i=1}^{m} [u_0, \gamma_i u_0] \sim [u_0, \gamma_1 \cdots \gamma_m u_0].$$

Now suppose that the permutation contains some cycle of order n, which means that σ contains a cycle

$$\sigma' = [a_1, \gamma_n a_n] + [a_2, \gamma_1 a_1] + [a_3, \gamma_2 a_2] + \cdots + [a_n, \gamma_{n-1} a_{n-1}].$$

Using 5. and 2. we can sum the first two terms

$$[a_1, \gamma_n a_n] + [a_2, \gamma_1 a_1] \sim [\gamma_1 a_1, \gamma_1 \gamma_n a_n] + [a_2, \gamma_1 a_1] \sim [a_2, \gamma_1 \gamma_n a_n].$$

By induction we then get $\sigma' \sim [a_n, \gamma_{n-1}\gamma_{n-2} \cdots \gamma_1 \gamma_n a_n]$, so that we are reduced to the first case. $\square$

Integral cohomology Let $R \subset \mathbb{C}$ be an $\mathscr{O}$-subalgebra. We have a pairing

$$\begin{aligned} \langle\ ,\ \rangle \colon H_1(Y_\Gamma;\mathbb{C}) \otimes H^1(Y_\Gamma;\mathbb{C}) &\longrightarrow \mathbb{C} \\ ([\sigma],[\omega]) &\longmapsto \int_\sigma \omega \end{aligned}$$

where $\omega \in \Omega^1(\mathbb{H}_3;\mathbb{C})^\Gamma$ and $\sigma \in C_1(\mathbb{H}_3)_\Gamma$; see [Fel05, Satz. 3]. Note that $R \otimes_R \mathbb{C} = \mathbb{C}$, so that we have a map

$$H_1(Y_\Gamma;R) \longrightarrow H_1(Y_\Gamma;R) \otimes_R \mathbb{C} = H_1(Y_\Gamma;\mathbb{C}).$$

Let $\widetilde{H}_1(Y_\Gamma;R)$ be the image of this map. The kernel is the torsion part of $H_1(Y_\Gamma;R)$, so that we can we identify $\widetilde{H}_1(Y_\Gamma;R)$ with the free part of $H_1(Y_\Gamma;R)$. We use the pairing to define the cohomology groups

$$\widetilde{H}^1(Y_\Gamma;R) \coloneqq \left\{\omega \in H^1(Y_\Gamma;\mathbb{C}) \,\middle|\, \langle \omega,\sigma\rangle \in R \quad \text{for all } \sigma \in \widetilde{H}_1(Y_\Gamma;R)\right\}.$$

Since R is torsion-free we can identify $H_1(Y_\Gamma;R)$ with $H_1(Y_\Gamma;\mathbb{Z}) \otimes_\mathbb{Z} R$. By Proposition 5.2.1 a class $[\omega]$ is in $\widetilde{H}^1(Y_\Gamma;R)$ if $\langle \omega, [u_0,\gamma u_0]\rangle \in R$ for all R.

Remark 5.2.1. This R-module is the torsion free part of the sheaf cohomology $H^1(Y_\Gamma;R)$, that we identify with the image

$$\operatorname{Im}\left(H^1(Y_\Gamma;R) \longrightarrow H^1(Y_\Gamma;\mathbb{C})\right).$$

5.2.3 Eisenstein map

In 5.2.2 we defined a restriction map

$$\operatorname{res}\colon H^1(Y_\Gamma;\mathbb{C}) \longrightarrow H^1(\partial X_\Gamma;\mathbb{C}).$$

The kernel is the interior cohomology $H^1_!(Y_\Gamma;\mathbb{C})$ and can be identified with the image of the compactly supported cohomology inside $H^1(Y_\Gamma;\mathbb{C})$. The complex conjugation sending $z+jv$ to $\bar{z}+jv$ induces a complex conjugation on $H^1(\mathcal{H}_\mathfrak{a}/\Gamma_\mathfrak{a};\mathbb{C}) \simeq H^1(\mathbb{C}/\mathfrak{a}^2;\mathbb{C})$, which is the same as the one induced from complex conjugation on $\mathbb{C}/\mathfrak{a}^2$. Let $H^1(\mathbb{C}/\mathfrak{a}^2;\mathbb{C})^\pm$ be the ±1-eigenspace, which is spanned by $dz \pm d\bar{z}$. Let

$$H^1(\partial X_\Gamma;\mathbb{C})^\pm \coloneqq \bigoplus_{[\mathfrak{a}]\in \mathrm{Cl}(K)} H^1(\mathbb{C}/\mathfrak{a}^2;\mathbb{C})^\pm.$$

Proposition 5.2.2. *We have*

$$\dim \operatorname{Im}(\operatorname{res}) = \frac{1}{2} \dim_{\mathbb{C}} H^1(\partial X_\Gamma; \mathbb{C}) = |Cl(K)|.$$

More precisely, $\operatorname{Im}(\operatorname{res})$ *is equal to the* -1*-eigenspace* $H^1(\partial X_\Gamma; \mathbb{C})^-$.

Proof. The result follows from a theorem of Serre, see [Ber09, Proposition. 24] for a proof. However let us prove the statement about the dimension. Let

$$\alpha^* \colon H^1(\partial X_\Gamma; \mathbb{C}) \longrightarrow H^2(X_\Gamma, \partial X_\Gamma; \mathbb{C})$$

be the map from the long exact sequence. By Poincaré duality we have

$$H^1(\partial X_\Gamma; \mathbb{C})^\vee \simeq H^1(\partial X_\Gamma; \mathbb{C})$$

and

$$H^2(X_\Gamma, \partial X_\Gamma; \mathbb{C}) \simeq H^2_c(Y_\Gamma; \mathbb{C}) \simeq H^1(Y_\Gamma; \mathbb{C})^\vee$$

so that we can see α^* as a map

$$\alpha^* \colon H^1(\partial X_\Gamma; \mathbb{C})^\vee \longrightarrow H^1(Y_\Gamma; \mathbb{C})^\vee$$

Since for $\theta \in \Omega^1(\partial X_\Gamma)$ and $\omega \in H^1(X_\Gamma, \partial X_\Gamma)$

$$\int_{X_\Gamma} \alpha(\theta) \wedge \omega = \int_{\partial X_\Gamma} \theta \wedge \operatorname{res}(\omega)$$

we have that $\alpha^* = \operatorname{res}^\vee$ is adjoint to res. It follows from

$$\operatorname{Im}(\operatorname{res}) = \ker(\alpha^*) = \ker(\operatorname{res}^\vee) = \operatorname{Im}(\operatorname{res})^\perp,$$

that $\operatorname{Im}(\operatorname{res})$ is an isotropic subspace, and thus it must be of half the dimension of the total space. □

For an unramified algebraic Hecke character ψ of infinity type z^2 we will define the Eisenstein map

$$\operatorname{Eis}_\psi \colon \operatorname{Im}(\operatorname{res}) \longrightarrow H^1(Y_\Gamma; \mathbb{C})$$

whose image is the Eisenstein cohomology $H^1_{\mathrm{Eis}}(Y_\Gamma; \mathbb{C})$. For an ideal $\mathfrak{a}$ we first define a map

$$\operatorname{Eis}_{\mathfrak{a}} \colon H^1(\mathbb{C}/\mathfrak{a}^2; \mathbb{C}) \longrightarrow H^1(Y_\Gamma; \mathbb{C}),$$

as follows. We have maps

$$\begin{aligned} p_{\mathfrak{b},\infty} \colon \Gamma(\mathfrak{b})_\infty \backslash \mathbb{H}_3 &\longrightarrow \mathbb{C}/\mathfrak{b}, \\ z + jv &\longmapsto z \end{aligned}$$

and

$$\begin{aligned} \Gamma_{\mathfrak{a}} \backslash \mathbb{H}_3 &\longrightarrow \Gamma(\mathfrak{a}^2)_\infty \backslash \mathbb{H}^3, \\ u &\longmapsto A_{\mathfrak{a}} u. \end{aligned}$$

Composing the two maps we get

$$p_{\mathfrak{a}} \colon \Gamma_{\mathfrak{a}} \backslash \mathbb{H}_3 \longrightarrow \mathbb{C}/\mathfrak{a}^2.$$

Now let $\omega \in \Omega^1(\mathbb{C}/\mathfrak{a}^2;\mathbb{C})$ and pull it back by $p_{\mathfrak{a}}$ to a form $p_{\mathfrak{a}}^*\omega \in \Omega^1(\mathbb{H}_3;\mathbb{C})^{\Gamma_{\mathfrak{a}}}$. In order to get a cohomology class on Y_Γ we could take the sum

$$\sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \gamma^* p_{\mathfrak{a}}^* \omega \in \Omega^1(\mathbb{H}_3;\mathbb{C})^\Gamma, \tag{5.2.1}$$

where we sum over a set of representatives of the quotient $\Gamma_{\mathfrak{a}} \backslash \Gamma$. However this series does not converge but can be regularized by adding a factor t^s for a complex parameter in s.

5.2.4 Regularization of the Eisenstein series

We will regularize the series (5.2.1). Since $p_{\mathfrak{a}}(u) = p_{\mathfrak{a}^2,\infty}(A_{\mathfrak{a}}u)$ we can rewrite (5.2.1) as

$$\sum_{\gamma \in \Gamma(\mathfrak{a}^2)_\infty \backslash A_{\mathfrak{a}}\Gamma} \gamma^* \omega \tag{5.2.2}$$

where here we simply write ω instead of $p_{\mathfrak{a}^2,\infty}^*\omega$. We have a bijection

$$\begin{aligned} \Gamma(\mathfrak{a}^2)_\infty \backslash A_{\mathfrak{a}}\Gamma &\longrightarrow \Gamma(m_{\mathfrak{a}} : n_{\mathfrak{a}}) \\ \gamma &\longmapsto \gamma^{-1}(1:0). \end{aligned}$$

Moreover, the orbit $\Gamma(m_{\mathfrak{a}} : n_{\mathfrak{a}})$ is in bijection with the set

$$\{ (m,n) \in \mathfrak{a} \times \mathfrak{a} | \gcd(m,n) = \mathfrak{a} \} / \mathscr{O}^\times$$

where $\mathscr{O}^\times = \{\pm 1\}$ acts diagonally on $\mathfrak{a} \times \mathfrak{a}$. Hence we get the bijection

$$\begin{aligned} \Gamma(\mathfrak{a}^2)_\infty \backslash A_\mathfrak{a}\Gamma &\longrightarrow \{(m,n) \in \mathfrak{a} \times \mathfrak{a} |\, \gcd(m,n) = \mathfrak{a}\} / \mathscr{O}^\times \\ \gamma = \begin{pmatrix} a & b \\ m & n \end{pmatrix} &\longmapsto (m,n). \end{aligned}$$

For $u = z + jv \in \mathbb{H}_3$ let $z(u) = z$, $\bar{z}(u) = \bar{z}$ and $v(u) = v$ be the coordinate functions. Let $\gamma \in A_\mathfrak{a}\Gamma$ and

$$\begin{aligned} \eta(m,n) &:= \gamma^*(dz) \\ \overline{\eta}(m,n) &:= \gamma^*(d\overline{z}) \end{aligned}$$

where $\gamma = \begin{pmatrix} a & b \\ m & n \end{pmatrix}$. The following calculations will indeed show that η and $\overline{\eta}$ depend only on m and n. We have

$$\begin{aligned} z(\gamma u) &= \frac{(az+b)\overline{(mz+n)} + a\overline{m}v^2}{|mz+n|^2 + |mv|^2}, \\ \overline{z}(\gamma u) &= \frac{\overline{(az+b)}(mz+n) + \overline{a}mv^2}{|mz+n|^2 + |mv|^2}. \end{aligned}$$

We view dz as the differential of the coordinate map $z(u)$, hence

$$\begin{aligned} \eta(m,n) &= \eta(m,n)_z dz + \eta(m,n)_{\bar{z}} d\bar{z} + \eta(m,n)_v dv \\ \overline{\eta}(m,n) &= \overline{\eta}(m,n)_z dz + \overline{\eta}(m,n)_{\bar{z}} d\bar{z} + \overline{\eta}(m,n)_v dv \end{aligned}$$

where

$$\begin{aligned} \eta(m,n)_z &= \frac{\partial z(\gamma u)}{\partial z} = \frac{\left(\overline{mz+n}\right)^2}{\left(|mz+n|^2 + |mv|^2\right)^2}, \\ \eta(m,n)_{\bar{z}} &= \frac{\partial z(\gamma u)}{\partial \bar{z}} = \frac{-(\overline{m}v)^2}{\left(|mz+n|^2 + |mv|^2\right)^2}, \\ \eta(m,n)_v &= \frac{\partial z(\gamma u)}{\partial v} = 2\frac{\left(\overline{mz+n}\right)\overline{m}v}{\left(|mz+n|^2 + |mv|^2\right)^2}, \end{aligned}$$

and

$$\overline{\eta}(m,n)_z = \frac{\partial \bar{z}(\gamma u)}{\partial z} = \frac{-(mv)^2}{\left(|mz+n|^2+|mv|^2\right)^2},$$
$$\overline{\eta}(m,n)_{\bar{z}} = \frac{\partial \bar{z}(\gamma u)}{\partial \bar{z}} = \frac{(mz+n)^2}{\left(|mz+n|^2+|mv|^2\right)^2},$$
$$\overline{\eta}(m,n)_v = \frac{\partial \bar{z}(\gamma u)}{\partial v} = 2\frac{(mz+n)\,mv}{\left(|mz+n|^2+|mv|^2\right)^2}.$$

It follows that when $\omega = dz$ we can rewrite (5.2.2)

$$\sum_{\gamma\in\Gamma(\mathfrak{a}^2)_\infty\backslash A_{\mathfrak{a}}\Gamma} \gamma^*(dz) = \frac{1}{2}\sum_{\substack{(m,n)\in\mathfrak{a}\times\mathfrak{a}\\ \gcd(m,n)=\mathfrak{a}}} \eta(m,n),$$

where the 2 comes from the assumption that $|\mathscr{O}^\times| = 2$. Similarly when $\omega = d\bar{z}$ we have

$$\sum_{\gamma\in\Gamma(\mathfrak{a}^2)_\infty\backslash A_{\mathfrak{a}}\Gamma} \gamma^*(d\bar{z}) = \frac{1}{2}\sum_{\substack{(m,n)\in\mathfrak{a}\times\mathfrak{a}\\ \gcd(m,n)=\mathfrak{a}}} \overline{\eta}(m,n).$$

Again this does not converge, hence for $s\in\mathbb{C}$ we define

$$\eta(m,n,s)_z := v^s\frac{\left(\overline{mz+n}\right)^2}{\left(|mz+n|^2+|mv|^2\right)^{2+s}},$$
$$\eta(m,n,s)_{\bar{z}} := v^s\frac{-(\overline{m}v)^2}{\left(|mz+n|^2+|mv|^2\right)^{2+s}},$$
$$\eta(m,n,s)_v := 2v^s\frac{\left(\overline{mz+n}\right)\overline{m}v}{\left(|mz+n|^2+|mv|^2\right)^{2+s}}.$$

We define $\overline{\eta}(m,n,s)_z, \overline{\eta}(m,n,s)_{\bar{z}}$ and $\overline{\eta}(m,n,s)_v$ similarly. For an ideal $\mathfrak{a}$ we define

$$\operatorname{Eis}_{\mathfrak{a}}(dz,s) := \frac{\mathrm{N}(\mathfrak{a})^s}{2}\sum_{\substack{(m,n)\in\mathfrak{a}\times\mathfrak{a}\\ \gcd(m,n)=\mathfrak{a}}} \eta(m,n,s)$$
$$\operatorname{Eis}_{\mathfrak{a}}(d\bar{z},s) := \frac{\mathrm{N}(\mathfrak{a})^s}{2}\sum_{\substack{(m,n)\in\mathfrak{a}\times\mathfrak{a}\\ \gcd(m,n)=\mathfrak{a}}} \overline{\eta}(m,n,s). \tag{5.2.3}$$

Note that since $\eta(\alpha m,\alpha n,s) = \alpha^{-2}\,\mathrm{N}(\alpha)^{-2s}\eta(m,n,s)$ we have

$$\operatorname{Eis}_{\alpha\mathfrak{a}}(dz,s) = \alpha^{-2}\,\mathrm{N}(\alpha)^{-s}\operatorname{Eis}_{\mathfrak{a}}(dz,s). \tag{5.2.4}$$

Let $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_h\}$ be a set of representatives for $\mathrm{Cl}(K)$. For an unramified Hecke character ψ of infinity type z^2 we define

$$\mathrm{Eis}_\psi(dz, s) := \sum_{i=1}^{h} \psi(\mathfrak{a}_i)^{-1} \mathrm{Eis}_{\mathfrak{a}_i}(dz, s)$$
$$\mathrm{Eis}_\psi(d\bar{z}, s) := \sum_{i=1}^{h} \psi(\mathfrak{a}_i)^{-1} \mathrm{Eis}_{\mathfrak{a}_i}(d\bar{z}, s).$$

On the other hand, Bergeron-Charollois-Garcia [BCG21] considered the forms

$$E_{\mathfrak{a}}(s) := \mathrm{N}(\mathfrak{a})^s \sum_{(m,n)\in\mathfrak{a}\times\mathfrak{a}} \eta(m, n, s),$$
$$\overline{E}_{\mathfrak{a}}(s) := \mathrm{N}(\mathfrak{a})^s \sum_{(m,n)\in\mathfrak{a}\times\mathfrak{a}} \eta(m, n, s).$$

Note that here we sum over $\mathfrak{a} \times \mathfrak{a}$ instead of the subset with $\gcd = \mathfrak{a}$ as is (5.2.3). After weighting with the character we also define the forms

$$E_\psi(s) := \sum_{i=1}^{h} \psi(\mathfrak{a}_i)^{-1} E_{\mathfrak{a}_i}(s),$$
$$\overline{E}_\psi(s) := \sum_{i=1}^{h} \psi(\mathfrak{a}_i)^{-1} \overline{E}_{\mathfrak{a}_i}(s).$$

Also note that $\mathrm{Eis}_\psi(dz, s)$ and $E_\psi(s)$ depend on the choice of representatives of the class group, but will see that their value at $s = 0$ do not. We have

$$E_{\mathfrak{a}}(s) = E_{\mathfrak{a},z}(s)dz + E_{\mathfrak{a},\bar{z}}(s)d\bar{z} + E_{\mathfrak{a},v}(s)dv$$

where

$$E_{\mathfrak{a},z}(s) := \mathrm{N}(\mathfrak{a})^s \sum_{(m,n)\in\mathfrak{a}\times\mathfrak{a}} \eta(m, n, s)_z,$$

and similarly for $E_{\mathfrak{a},\bar{z}}(s)$ and $E_{\mathfrak{a},v}(s)$.

Proposition 5.2.3. *The series*

$$E_{\mathfrak{a}}(s),\ \mathrm{Eis}_{\mathfrak{a}}(dz, s),\ \overline{E_{\mathfrak{a}}}(s), \mathrm{Eis}_{\mathfrak{a}}(d\bar{z}, s)$$

converge for $\mathrm{Re}(s) \gg 0$ *and admit an analytic continuation to the whole plane. We denote the* *various values at $s=0$ by*

$$E_{\mathfrak{a}}, \overline{E}_{\mathfrak{a}}, E_{\psi}, \overline{E}_{\psi}, \operatorname{Eis}_{\mathfrak{a}}(dz), \operatorname{Eis}_{\mathfrak{a}}(d\overline{z}), \operatorname{Eis}_{\psi}(dz), \operatorname{Eis}_{\psi}(d\overline{z}).$$

Moreover, the forms are closed at $s=0$.

Proof. The analytic continuation can be done by Poisson summation, see for example [BCG21, page. 18]. The fact that the forms $E_{\mathfrak{a}}(s)$ and $\overline{E_{\mathfrak{a}}}(s)$ are closed is the content of [BCG21, Proposition. 3.3]. □

The cohomology $H^1(\mathbb{C}/\mathfrak{a}^2;\mathbb{C})$ is generated by dz and $d\bar{z}$. Hence we extend $\operatorname{Eis}_{\mathfrak{a}}$ by linearity to a map

$$\begin{aligned} \operatorname{Eis}_{\mathfrak{a}} \colon H^1(\mathbb{C}/\mathfrak{a}^2;\mathbb{C}) &\longrightarrow H^1(Y_\Gamma,\mathbb{C}) \\ \omega = \alpha dz + \beta d\overline{z} &\longmapsto \operatorname{Eis}_{\mathfrak{a}}(\omega) := \alpha \operatorname{Eis}_{\mathfrak{a}}(dz) + \beta \operatorname{Eis}_{\mathfrak{a}}(d\overline{z}). \end{aligned}$$

Since $H^1(\partial X_\Gamma;\mathbb{C}) = \bigoplus_{[\mathfrak{a}]\in \mathrm{Cl}(K)} H^1(\mathbb{C}/\mathfrak{a}^2;\mathbb{C})$ we then have a map

$$\begin{aligned} \operatorname{Eis}_{\psi} \colon H^1(\partial X_\Gamma;\mathbb{C}) &\longrightarrow H^1(Y_\Gamma;\mathbb{C}) \\ (\omega^{(\mathfrak{a})})_{[\mathfrak{a}]\in \mathrm{Cl}(K)} &\longmapsto \sum_{[\mathfrak{a}]\in \mathrm{Cl}(K)} \psi(\mathfrak{a})^{-1} \operatorname{Eis}_{\mathfrak{a}}(\omega^{(\mathfrak{a})}), \end{aligned}$$

where ψ is as above. In particular we can restrict it to a map

$$\operatorname{Eis}_{\psi} \colon \operatorname{Im}(\mathrm{res}) \longrightarrow H^1(Y_\Gamma;\mathbb{C}).$$

Since $\psi((\alpha)) = \alpha^{-2}$ and by (5.2.4) the inner sum in 5.2.4 only depends on the ideal class and $\operatorname{Eis}_{\psi}$ does not depend on the choice of representatives.

Lemma 5.2.4. *We have $E_{\psi} = 2L(\psi,0)\operatorname{Eis}_{\psi}(dz)$ and $\overline{E}_{\psi} = 2L(\psi,0)\operatorname{Eis}_{\psi}(d\bar{z})$.*

Proof. Since

$$2\operatorname{Eis}_{\mathfrak{a}}(dz,s) = \mathrm{N}(\mathfrak{a})^s \sum_{\substack{(m,n)\in \mathfrak{a}\times\mathfrak{a} \\ \gcd(m,n)=\mathfrak{a}}} \eta(m,n,s)$$

we have

$$\begin{aligned} E_{\mathfrak{a}}(s) &= \mathrm{N}(\mathfrak{a})^s \sum_{(m,n)\in\mathfrak{a}\times\mathfrak{a}} \eta(m,n,s) \\ &= \mathrm{N}(\mathfrak{a})^s \sum_{0\neq\mathfrak{b}\subseteq\mathscr{O}} \sum_{\substack{(m,n)\in\mathfrak{a}\times\mathfrak{a}\\ \gcd(m,n)=\mathfrak{a}\mathfrak{b}}} \eta(m,n,s) \\ &= 2\sum_{0\neq\mathfrak{b}\subseteq\mathscr{O}} \mathrm{N}(\mathfrak{b})^{-s}\,\mathrm{Eis}_{\mathfrak{b}\mathfrak{a}}(dz,s). \end{aligned}$$

Thus

$$\begin{aligned} E_\psi(s) &= \sum_{i=1}^{h} \psi(\mathfrak{a}_i)^{-1} E_{\mathfrak{a}_i}(s) \\ &= 2\sum_{0\neq\mathfrak{b}\subseteq\mathscr{O}} \mathrm{N}(\mathfrak{b})^{-s} \sum_{i=1}^{h} \psi(\mathfrak{a}_i)^{-1}\,\mathrm{Eis}_{\mathfrak{b}\mathfrak{a}_i}(s). \end{aligned}$$

After replacing the representatives $\{\widetilde{\mathfrak{a}_1},\dots,\widetilde{\mathfrak{a}_h}\} = \{\mathfrak{b}\mathfrak{a}_1,\dots,\mathfrak{b}\mathfrak{a}_h\}$ and we have

$$\begin{aligned} E_\psi(s) &= 2\sum_{0\neq\mathfrak{b}\subseteq\mathscr{O}} \psi(\mathfrak{b})\,\mathrm{N}(\mathfrak{b})^{-s} \sum_{i=1}^{h} \psi(\widetilde{\mathfrak{a}_i})^{-1}\,\mathrm{Eis}_{\widetilde{\mathfrak{a}_i}}(dz,s) \\ &= 2L(\psi,s)\,\mathrm{Eis}_\psi(dz,s). \end{aligned}$$

□

5.2.5 Relation to the Sczech cocycle

The forms $E_{\mathfrak{a}}$ and $\overline{E}_{\mathfrak{a}} \in \Omega^1(Y_\Gamma;\mathbb{C})$ define cocycles in $H^1(\Gamma;\mathbb{C})$

$$E_{\mathfrak{a}}(\gamma) \coloneqq \int_{u_0}^{\gamma u_0} E_{\mathfrak{a}}, \qquad \overline{E}_{\mathfrak{a}}(\gamma) \coloneqq \int_{u_0}^{\gamma u_0} \overline{E}_{\mathfrak{a}}.$$

Since the forms are closed the integrals do not depend on the path from u_0 to γu_0. By summing over the different cusps we get cocycles

$$E_\psi \coloneqq \sum_{[\mathfrak{a}]\in\mathrm{Cl}(K)} \psi(\mathfrak{a})^{-1} E_{\mathfrak{a}}, \qquad \overline{E}_\psi \coloneqq \sum_{[\mathfrak{a}]\in\mathrm{Cl}(K)} \psi(\mathfrak{a})^{-1} \overline{E}_{\mathfrak{a}}.$$

Theorem 5.2.5. *We have*

$$E_\psi = \sum_{[\mathfrak{a}]\in Cl(K)} \psi(\mathfrak{a})^{-1}\Phi_{\mathfrak{a}}, \qquad \overline{E}_\psi = -\sum_{[\mathfrak{a}]\in Cl(K)} \psi(\mathfrak{a})^{-1}\Phi_{\mathfrak{a}}.$$

Proof. It is enough to show that $E_{\mathfrak{a}}(\gamma) = \Phi_{\mathfrak{a}}(\gamma)$. Let $E^*_{\mathfrak{a}}$ be the extension of $E_{\mathfrak{a}}$ to $\mathbb{H}_3^*$, defined by setting $E_{\mathfrak{a}}^{(r)} := \lim_{z\to r} E_{\mathfrak{a}}$. When $r = \infty$ then $E_{\mathfrak{a}}^{(\infty)}$ is the constant term at the cusp ∞. For $E_{\mathfrak{a},z}(s)$ it is given by

$$\begin{aligned} E_{\mathfrak{a},z}^{(\infty)}(s) &= \mathrm{N}(\mathfrak{a})^s \sum_{n\in\mathfrak{a}} \eta(0,n,s) \\ &= v^s\,\mathrm{N}(\mathfrak{a})^s \sum_{n\in\mathfrak{a}} \frac{\bar{n}^2}{|n|^{2s+4}} \\ &= v^s\,\mathrm{N}(\mathfrak{a})^s G(s+1,2,0,0,\mathfrak{a}). \end{aligned}$$

Similarly (see the formulas in [Ito87, p.154]) we have

$$E_{\mathfrak{a},\bar{z}}^{(\infty)}(s) = -v^s \frac{2i\pi}{(s+1)D(\mathfrak{a})} G(s,2,0,0,\mathfrak{a}) + ...$$

and

$$\begin{aligned} \overline{E}_{\mathfrak{a},z}^{(\infty)}(s) &= -v^s \frac{2i\pi}{(s+1)D(\mathfrak{a})} G(s,2,0,0,\mathfrak{a}) \\ \overline{E}_{\mathfrak{a},\bar{z}}^{(\infty)}(s) &= v^s G(s+1,2,0,0,\mathfrak{a}). \end{aligned}$$

Hence, at $s = 0$, the constant term is given by

$$\begin{aligned} E_{\mathfrak{a}}^{(\infty)} &= G_2(\mathfrak{a})dz - G(\mathfrak{a})d\bar{z}, \\ \overline{E}_{\mathfrak{a}}^{(\infty)} &= -G(\mathfrak{a})dz + G_2(\mathfrak{a})d\bar{z} = -E_{\mathfrak{a}}^{(\infty)}. \end{aligned}$$

Moreover, by the functional equation (5.1.2) we have

$$G_2(\mathfrak{a}) = G(\mathfrak{a}).$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

1. Suppose that $c \neq 0$. Let $P(\gamma)$ be the closed path in $\mathbb{H}_3^*$ pictured in Figure 5.3. Since the

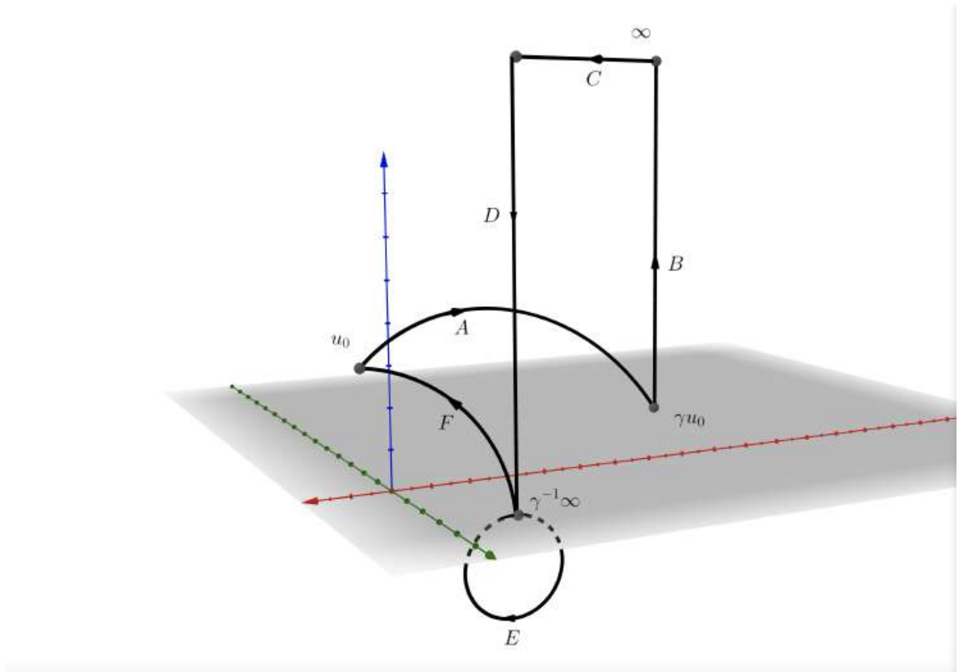


Figure 5.3: The path $P(\gamma)$ when $c \neq 0$

path is closed we have

$$\int_{P(\gamma)} E_{\mathfrak{a}}^* = 0.$$

Note that E is an arc in the horocycle $\mathcal{H}_{\gamma^{-1}\infty}$ and C is a segment in $\mathcal{H}_\infty$. Let $\widetilde{D} \coloneqq \gamma D$, $\widetilde{E} \coloneqq \gamma E$, $\widetilde{F} \coloneqq \gamma F$ and $\widetilde{P}(\gamma)$ be the path in Figure 5.4. Since $E_{\mathfrak{a}}$ is Γ-invariant we have

$$\int_{\widetilde{P}(\gamma)} E_{\mathfrak{a}}^* = \int_{P(\gamma)} E_{\mathfrak{a}}^* = 0.$$

Moreover, since the integrals along B and $\widetilde{F}$ cancel we get

$$\int_{u_0}^{\gamma u_0} E_{\mathfrak{a}} = -\int_{C+\widetilde{E}} E_{\mathfrak{a}}^{(\infty)} - \int_{\widetilde{D}} E_{\mathfrak{a}}.$$

We will compute these two integrals separately.

(a) We begin with integral along $C + \widetilde{E}$. Let $z(t) = \frac{a}{c} - t\frac{a+d}{c}$ with $t \in [0,1]$ be the straight line joining the endpoints of $\widetilde{E} + C$, where $\gamma\infty = \frac{a}{c}$ and $\gamma^{-1}\infty = -\frac{d}{c}$. Since $E_{\mathfrak{a}}^{(\infty)}$ is

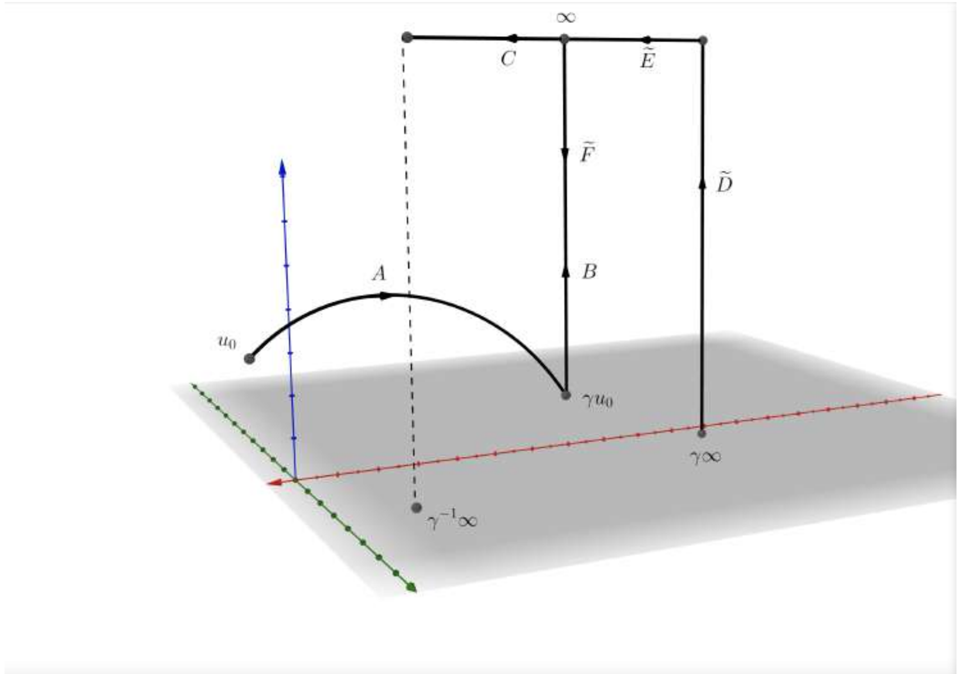


Figure 5.4: The path $\widetilde{P}(\gamma)$ is Γ-equivalent to the path $P(\gamma)$ in Figure 5.3

closed the integral along $\widetilde{E}+C$ is the same as along the line $z(t)$. Then

$$\begin{aligned}\int_{C+\widetilde{E}} E_{\mathfrak{a}}^{(\infty)} &= G_2(\mathfrak{a})\int_0^1 (dz(t) - d\bar{z}(t)) \\ &= -G_2(\mathfrak{a}) I\left(\frac{a+d}{c}\right)\end{aligned}$$

and

$$\int_{C+\widetilde{E}} \overline{E}_{\mathfrak{a}}^{(\infty)} = G_2(\mathfrak{a}) I\left(\frac{a+d}{c}\right).$$

(b) Let us now compute the integral along $\widetilde{D}$. The path $\widetilde{D}$ is parametrized by $u(t) = \frac{a}{c} + jt$ with $0 < t < \infty$. Hence we have

$$\begin{aligned}\int_{\widetilde{D}} E_{\mathfrak{a}}(s) &= \int_{\widetilde{D}} E_{\mathfrak{a},v}(s) \\ &= 2\sum_{(m,n)\in\mathfrak{a}\times\mathfrak{a}} \int_0^\infty \frac{\left(\overline{m\frac{a}{c}+n}\right)\overline{m}t^{1+s}}{\left(|m\frac{a}{c}+n|^2 + |m|^2t^2\right)^{2+s}} dt \\ &= 2\sum_{(m,n)\in\mathfrak{a}\times\mathfrak{a}} \frac{\left(\overline{m\frac{a}{c}+n}\right)\overline{m}}{|m\frac{a}{c}+n|^{2(2+s)}} \int_0^\infty \frac{t^{1+s}}{\left(1+\frac{|m|^2}{|m\frac{a}{c}+n|^2}t^2\right)^{2+s}} dt.\end{aligned}$$

By substituting $\alpha = \frac{|m|}{|m\frac{a}{c}+n|}t$ this becomes

$$
\begin{aligned}
2 \sum_{(m,n)\in\mathfrak{a}\times\mathfrak{a}} \frac{\left(\overline{m\frac{a}{c}+n}\right)}{|m\frac{a}{c}+n|^{2+s}} \frac{\overline{m}}{|m|^{2+s}} \int_0^\infty \frac{\alpha^{1+s}}{(1+\alpha^2)^{2+s}} d\alpha \\
= B\left(1+\frac{s}{2}, 1+\frac{s}{2}\right) \sum_{(m,n)\in\mathfrak{a}\times\mathfrak{a}} \frac{\left(\overline{m\frac{a}{c}+n}\right)}{|m\frac{a}{c}+n|^{2+s}} \frac{\overline{m}}{|m|^{2+s}},
\end{aligned}
$$

where

$$
B(x,y) = \int_0^\infty \frac{t^{y-1}}{(1+t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
$$

is the Beta function. By writing $m = c\widetilde{m} + r$ and summing over $\widetilde{m}$ and r we rewrite the previous sum as

$$
B\left(1+\frac{s}{2}, 1+\frac{s}{2}\right) \sum_{r\in\mathfrak{a}/c\mathfrak{a}} \sum_{(\widetilde{m},n)\in\mathfrak{a}\times\mathfrak{a}} \frac{\left(\overline{r\frac{a}{c}+a\widetilde{m}+n}\right)}{|r\frac{a}{c}+a\widetilde{m}+n|^{2+s}} \frac{\overline{c\widetilde{m}+r}}{|c\widetilde{m}+r|^{2+s}}.
$$

By summing over $(\widetilde{m}, a\widetilde{m}+n)$ instead of $(\widetilde{m}, n)$ we rewrite the sums

$$
B\left(1+\frac{s}{2}, 1+\frac{s}{2}\right) \sum_{r\in\mathfrak{a}/c\mathfrak{a}} \frac{1}{c|c|^s} G_1\left(\frac{1+s}{2}, 1, \frac{ar}{c}, 0,\right) G_1\left(\frac{1+s}{2}, 1, \frac{r}{c}, 0,\right).
$$

Hence at $s = 0$ we get

$$
\int_{\widetilde{D}} E_{\mathfrak{a}} = D(a, c, \mathfrak{a}),
$$

and by a similar computation also

$$
\int_{\widetilde{D}} \overline{E}_{\mathfrak{a}} = D(\overline{a}, \overline{c}, \mathfrak{a}) = -D(a, c, \mathfrak{a}).
$$

2. Suppose that $c = 0$. Let $P(\gamma)$ the closed path pictured in Figure 5.5. The integrals along the paths B and D in Figure 5.5 cancel, and we have

$$
\int_{u_0}^{\gamma u_0} E_{\mathfrak{a}} = -\int_C E_{\mathfrak{a}}^{(\infty)}.
$$

Since K has no non-trivial units we have $a, d = \pm 1$ and $a/d = 1$. If $u_0 = z_0 + jv_0$ then $\gamma u_0 = z_0 + \frac{b}{d} + jv_0$. Hence the path C is parametrized by $z(t) = z_0 + \frac{b}{d}(1-t)$ for $t \in [0,1]$.

At the boundary the form is

$$E_{\mathfrak{a}}^{(\infty)} = G_2(\mathfrak{a})\,(dz - d\bar{z})$$

and thus

$$\int_C E_{\mathfrak{a}}^{(\infty)} = -G_2(\mathfrak{a}) I \left(\frac{b}{d}\right).$$

Similarly we get

$$\int_C \overline{E}_{\mathfrak{a}}^{(\infty)} = G_2(\mathfrak{a}) I \left(\frac{b}{d}\right).$$

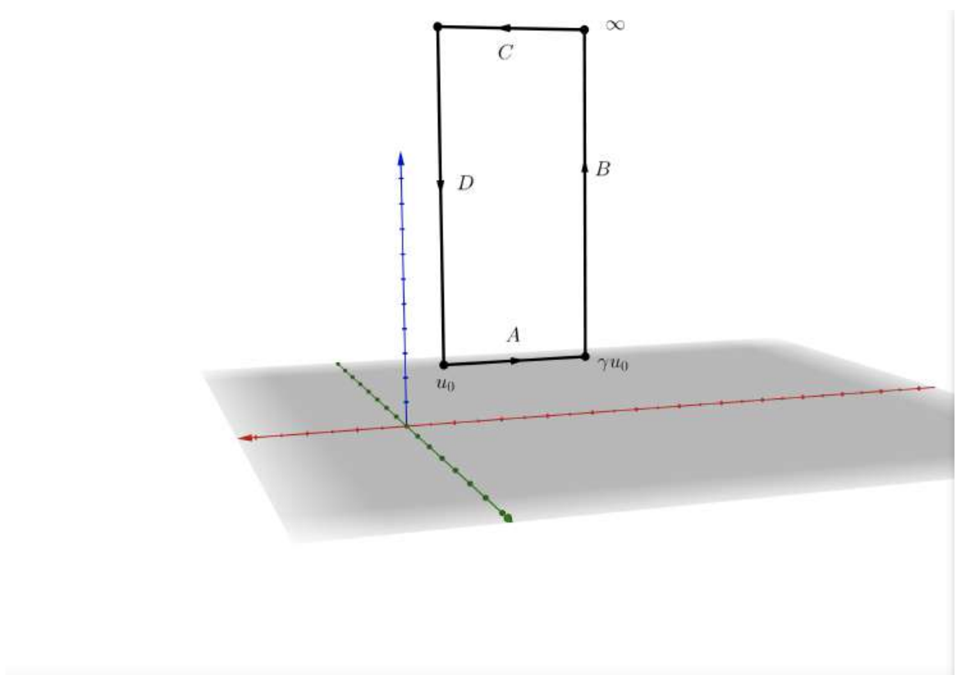


Figure 5.5: The path $P(\gamma)$ for $c = 0$

□

5.3 Denominators of the Eisenstein cohomology

Let K_ψ be an extension of K containing the image of Φ_L, the value $L(\psi, 0)$ and the field F_ψ. Let $\mathscr{O}_\psi$ be the ring of integers of K_ψ. We fix an embedding $\sigma\colon \overline{K} \longrightarrow \mathbb{C}$, so that we view $\mathscr{O}_\psi$ as an

$\mathscr{O}$-subalgebra of $\mathbb{C}$. One would like to know if the Eisenstein map preserves the rational and integral structures. Harder showed that it preserves the rational structure, so that we have a map

$$\mathrm{Eis}_\psi\colon\ \mathrm{Im}(\mathrm{res}) \cap H^1(\partial X_\Gamma; K_\psi) \longrightarrow H^1(Y_\Gamma; K_\psi).$$

However, in general the map Eis_ψ is not integral, in the sense that the image of

$$\mathrm{Eis}_\psi\colon\ \mathrm{Im}(\mathrm{res}) \cap \widetilde{H}^1(\partial X_\Gamma; \mathscr{O}_\psi) \longrightarrow H^1(Y_\Gamma; K_\psi)$$

does not lie in $\widetilde{H}^1(Y_\Gamma; \mathscr{O}_\psi)$. We define the denominator of $c \in H^1(Y_\Gamma; K_\psi)$ to be the fractional ideal

$$\delta_{K_\psi}(c) \coloneqq \left\{ a \in K_\psi \,|\, ac \in \widetilde{H}^1(Y_\Gamma; \mathscr{O}_\psi) \right\}.$$

We also define the denominator of the Eisenstein cohomology

$$\delta_{K_\psi}(\mathrm{Eis}_\psi) \coloneqq \bigcap_{[\omega]\in \mathrm{Im}(\mathrm{res})\cap \widetilde{H}^1(\partial X_\Gamma; \mathscr{O}_\psi)} \delta_{K_\psi}(\mathrm{Eis}_\psi(\omega)). \tag{5.3.1}$$

If the denominator were $\mathscr{O}_\psi$ then the Eisenstein map would be integral, so the denominator measures how far from integral it is. Berger [Ber09] proves in many cases a lower bound (in the sense of divisibility) on the denominator. We will prove an upper bound in the case where the class number of K is one.

5.3.1 Integrality of Eisenstein series

We begin by proving some integrality results of Eisenstein series, due to Damerell [Dam71]. We will need the following result, due to Cassels [Cas49].

Proposition 5.3.1. *Let n be a positive integer such that nz is in L. Then $2\sqrt{n}\wp(z)$ is an algebraic integer.*

Proposition 5.3.2. *Let L be a lattice with $\mathscr{O}(L) = \mathscr{O}$ and such that $g_2(L)$ and $g_3(L)$ are algebraic integers. If nz is in L, then the values $2^{\frac{1}{2}} n^{\frac{5}{4}} G_1(z, L)$ and $2^{\frac{1}{2}}\sqrt{d_K} G_2(L)$ are algebraic integers.*

Proof. We follow Sczech's presentation [Scz86, Proof of Theorem. 10] of Damerell's proof, and begin with the integrality of $G_1(z, L)$. First note that if $z_0 \in L$, then $G_1(z + z_0, L) = G_1(z, L)$ and that

$G_1(-z,L) = -G_1(z,L)$. Hence if z is such that $nz \in L$, we have

$$\begin{aligned} nG_1(z,L) &= (n-1)G_1(z,L) + G_1(z,L) \\ &= (n-1)G_1(z,L) + G_1(nz-(n-1)z,L) \\ &= (n-1)G_1(z,L) - G_1((n-1)z,L). \end{aligned}$$

We can write this as a telescopic sum

$$nG_1(z,L) = \sum_{k=1}^{n-1} G_1(kz,L) + G_1(z,L) - G_1((k+1)z,L). \tag{5.3.2}$$

For three points x, y, w in L with $x+y+w=0$ we have the addition formula

$$(G_1(x,L) + G_1(y,L) + G_1(w,L))^2 = \wp(x) + \wp(y) + \wp(w).$$

If $nz \in L$ then by the result of Cassels

$$2^{\frac{1}{2}} n^{\frac{1}{4}} \left(G_1(kz,L) + G_1(z,L) - G_1((k+1)z,L)\right)$$

is an algebraic integer, where we used the fact that the square root of an algebraic integer is an algebraic integer. Hence by (5.3.2) we have that

$$2^{\frac{1}{2}} n^{\frac{5}{4}} G_1(z,L)$$

is an algebraic integer. For the integrality of $G_2(L)$, note that we have

$$\wp(z) = G_2(z,L) - G_2(L).$$

Hence for any non-real $\alpha \in \mathscr{O} - \{0\}$ we have

$$\sum_{r \in L/\alpha L} \wp\left(\frac{r}{\alpha}\right) = \sum_{r \in L/\alpha L} \left(G_2\left(\frac{r}{\alpha}, L\right) - G_2(L)\right) = (\alpha^2 - \alpha\overline{\alpha})G_2(L).$$

We apply Cassel's result with $n := (\alpha\overline{\alpha})^2 = \mathrm{N}(\alpha)^2 \in \mathbb{N}$, since $\mathscr{O}(L) = \mathscr{O}$ we have

$$n\frac{r}{\alpha} = \mathrm{N}(\alpha)\overline{\alpha}r \in L.$$

We write $\alpha = x_\alpha + y_\alpha \frac{d_K + i\sqrt{|d_K|}}{2}$. By Cassel's result we have that $2^{\frac{1}{2}}\,\mathrm{N}(\alpha)\alpha i y_\alpha \sqrt{|d_K|} G_2(L)$ is an algebraic integer. Hence after multiplying by the algebraic integer $-i\overline{\alpha}$ we also have that

$$2^{\frac{1}{2}}\,\mathrm{N}(\alpha)^2 y_\alpha \sqrt{|d_K|}G_2(L)$$

is an algebraic integer. Finally we choose α and α' such that $\mathrm{N}(\alpha)^2 y_\alpha$ and $\mathrm{N}(\alpha')^2 y_{\alpha'}$ are coprime and we find that

$$2^{\frac{1}{2}}\sqrt{|d_K|}G_2(L)$$

is an algebraic integer. □

5.3.2 Integrality of the Sczech cocycle

Since $E_\psi(\gamma) = \int_{u_0}^{\gamma u_0} E_\psi$ we have

$$E_\psi(\gamma_1\gamma_2) = E_\psi(\gamma_1) + E_\psi(\gamma_2)$$

and it follows from Theorem 5.2.5 that

$$\Phi_L(\gamma_1\gamma_2) = \Phi_L(\gamma_1) + \Phi_L(\gamma_2). \tag{5.3.3}$$

This gives an alternative proof of the cocycle property of the Sczech cocycle.

Lemma 5.3.3. *Let a and c in $\mathscr{O} - 0$ be coprime. Then there exists an $x \in \mathscr{O}$ such that $|a + cx|^2$ and $|c|^2$ are coprime.*

Proof. First note that for an integer $n \in \mathbb{Z}$ we have $\gcd(n, c) = 1$ implies $\gcd(n, |c|^2) = 1$, or equivalently $\gcd(n, |c|^2) > 1$ implies $\gcd(n, c) > 1$. Indeed, if $\mathfrak{p}$ is a prime ideal dividing $\gcd(n, |c|^2)$ but not dividing c then $\mathfrak{p}$ divides $\gcd(n, \overline{c})$. On the other hand, since n is an integer, if $\mathfrak{p}$ divides n then $\overline{\mathfrak{p}}$ divides n. It follows that $\overline{\mathfrak{p}}$ divides $\gcd(n, c)$. Hence we only need to find x such that $\gcd(|a + cx|^2, c) = 1$.

For an ideal $\mathfrak{a}$ let $S(\mathfrak{a})$ be the set of its prime ideal divisors. We write $S(c)$ as a disjoint union

$$S(c) = S_1 \sqcup S_2 \sqcup S_3$$

where

$$\begin{aligned}
S_1 &\coloneqq S(c) \cap S(\overline{a}) = \{\mathfrak{p} \text{ divides } c \text{ and } \overline{a}\}\\
S_2 &\coloneqq S(c) \cap S(\overline{c}) = \{\mathfrak{p} \text{ divides } c \text{ and } \overline{c}\}\\
S_3 &\coloneqq S(c) - S(c) \cap S(\overline{ac}) = \{\mathfrak{p} \text{ divides } c \text{ and does not divide } \overline{ac}\}.
\end{aligned}$$

Note that since $\overline{a}, \overline{c}$ are coprime $S_1 \cap S_2 = \emptyset$. By the Chinese remainder theorem we can find $x \in \mathscr{O}$ such that

$$\begin{aligned} v_{\mathfrak{p}}(\overline{x}) &= 0 \qquad \text{for all } \mathfrak{p} \in S_1, \\ v_{\mathfrak{p}}(\overline{x}) &> 0 \qquad \text{for all } \mathfrak{p} \in S_3. \end{aligned}$$

Let us now show that $v_{\mathfrak{p}}(|a+cx|^2) = 0$ for all $\mathfrak{p} \in S(c)$. Since $\mathfrak{p}$ divides c we have $v_{\mathfrak{p}}(c) > 0$ and $v_{\mathfrak{p}}(a) = 0$, and because $v_{\mathfrak{p}}(x) \geq 0$ it follows that

$$v_{\mathfrak{p}}(a+cx) = \min(v_{\mathfrak{p}}(a), v_{\mathfrak{p}}(c) + v_{\mathfrak{p}}(x)) = 0.$$

It remains to show that $v_{\mathfrak{p}}(\overline{a+cx}) = 0$.

1. If $\mathfrak{p} \in S_1$ we have $v_{\mathfrak{p}}(\overline{a}) > 0$ and $v_{\mathfrak{p}}(\overline{c}) = v_{\mathfrak{p}}(\overline{x}) = 0$ so that
$$v_{\mathfrak{p}}(\overline{a+cx}) = \min(v_{\mathfrak{p}}(\overline{a}), v_{\mathfrak{p}}(\overline{c}) + v_{\mathfrak{p}}(\overline{x})) = 0.$$
2. If $\mathfrak{p} \in S_2$ we have $v_{\mathfrak{p}}(\overline{c}) > 0$, $v_{\mathfrak{p}}(\overline{a}) = 0$ and $v_{\mathfrak{p}}(\overline{x}) \geq 0$ so that
$$v_{\mathfrak{p}}(\overline{a+cx}) = \min(v_{\mathfrak{p}}(\overline{a}), v_{\mathfrak{p}}(\overline{c}) + v_{\mathfrak{p}}(\overline{x})) = 0.$$
3. Finally if $\mathfrak{p} \in S_3$ we have $v_{\mathfrak{p}}(\overline{c}) = v_{\mathfrak{p}}(\overline{a}) = 0$ and $v_{\mathfrak{p}}(\overline{x}) > 0$ so that we also have
$$v_{\mathfrak{p}}(\overline{a+cx}) = \min(v_{\mathfrak{p}}(\overline{a}), v_{\mathfrak{p}}(\overline{c}) + v_{\mathfrak{p}}(\overline{x})) = 0.$$

□

Proposition 5.3.4 (Sczech)**.** *Let $L \subset K$ be a lattice such that $g_2(L)$ and $g_3(L)$ are integers. Then $2\Phi_L$ is an algebraic integer.*

Proof. It follows from Proposition 5.3.2, we recall the proof from Sczech [Scz84, Satz. 4]. Note that by the definition of the Sczech cocycle we have $\Phi_L(A) = 0$ for $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence by (5.3.3) we have

$$\Phi_L \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Phi_L \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \Phi_L \begin{pmatrix} c & d \\ -a & -b \end{pmatrix} \qquad (5.3.4)$$

First suppose that $c = 0$. Then a and d are units and

$$I\left(\frac{b}{d}\right) \in \sqrt{d_K}\mathbb{Z}.$$

Hence it follows from Proposition 5.3.2 that

$$2\Phi_L \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \tag{5.3.5}$$

is integral.

Suppose that $c \neq 0$. By (5.3.4) we can also suppose that $a \neq 0$. Since $\frac{ar}{c}$ and $\frac{r}{c}$ are $|c|^2$-torsion points we have

$$2|c|^7 D(a, c, L)$$

is an algebraic integer. Since $I\left(\frac{a+d}{c}\right) \in \sqrt{d_K}\frac{1}{|c|^2}\mathbb{Z}$ we have that

$$2|c|^7 \Phi_L \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is integral. By (5.3.4) we also have that

$$2|a|^7 \Phi_L \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is integral. Hence if $|a|^2$ and $|c|^2$ are coprime, then

$$2\Phi_L \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.3.6}$$

is integral. If not then by Lemma 5.3.3 we can find $x \in \mathscr{O}$ such that $|a + cx|^2$ and $|c|^2$ are coprime. Since

$$\begin{pmatrix} a + cx & b + dx \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have by (5.3.3)

$$\Phi_L \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Phi_L \begin{pmatrix} a + cx & b + dx \\ c & d \end{pmatrix} - \Phi_L \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

The integrality follows from the integrality of the terms on the right-hand side, which in turn follows from (5.3.5) and (5.3.6). □

5.3.3 An upper bound on the denominator

Theorem 5.3.5. *Suppose that K has class number 1. An upper bound for the denominator of the Eisenstein cohomology is given by*

$$2L^{\mathrm{alg}}(\psi,0)\mathscr{O}_\psi \subset \delta_{K_\psi}(\mathrm{Eis}_\psi).$$

Proof. Recall that by Proposition 5.2.1 a class $[\omega]$ is in $\widetilde{H}^1(Y_\Gamma;\mathscr{O}_\psi)$ if $\langle \omega, [u_0,\gamma u_0]\rangle \in R$ for every $\gamma\in\Gamma$. Let $\omega \in \mathrm{Im}(\mathrm{res})\cap \widetilde{H}^1(\partial X_\Gamma,\mathscr{O}_\psi)$, that we can write $\omega=\alpha(dz-d\bar{z})$ with $\alpha\in\mathscr{O}_\psi$. It follows from Lemma 5.2.4 that

$$E_\psi(\gamma)=\int_{u_0}^{\gamma u_0} E_\psi = \big\langle E_\psi,[u_0,\gamma u_0]\big\rangle = \big\langle 2L(\psi,0)\,\mathrm{Eis}_\psi(dz),[u_0,\gamma u_0]\big\rangle.$$

Hence $\big\langle 2L^{\mathrm{alg}}(\psi,0)\,\mathrm{Eis}_\psi(\omega),[u_0,\gamma u_0]\big\rangle = \alpha\Omega^{-2}\left(E_\psi(\gamma)-\overline{E}_\psi(\gamma)\right)$. By Propositions 5.2.5 and 5.3.4 we have

$$\alpha\Omega^{-2}\left(E_\psi(\gamma)-\overline{E}_\psi(\gamma)\right)=2\alpha\Omega^{-2}\Phi_{\mathscr{O}}(\gamma)=2\alpha\Phi_L(\gamma)\in\mathscr{O}_\psi.$$

Thus for any $\omega\in\mathrm{Im}(\mathrm{res})\cap\widetilde{H}^1(\partial X_\Gamma,\mathscr{O}_\psi)$ we have

$$2L^{\mathrm{alg}}(\psi,0)\,\mathrm{Eis}_\psi(\omega)\in\widetilde{H}^1(Y_\Gamma,\mathscr{O}_\psi).$$

□

5.3.4 Relation to the work of Berger

Instead of the fractional ideal $\delta_{K_\psi}(\mathrm{Eis}_\psi)$ defined in (5.3.1), we can also consider the integral denominator

$$\delta_{\mathscr{O}_\psi}(c) \coloneqq \left\{a\in\mathscr{O}_\psi \,|\, ac\in\widetilde{H}^1(Y_\Gamma;\mathscr{O}_\psi)\right\}\subset\mathscr{O}_\psi,$$

and

$$\delta_{\mathscr{O}_\psi}(\mathrm{Eis}_\psi)=\bigcap_{\omega\in\mathrm{Im}(\mathrm{res})\cap\widetilde{H}^1(\partial X_\Gamma;\mathscr{O}_\psi)}\delta_{\mathscr{O}_\psi}(\mathrm{Eis}_\psi(\omega)).$$

By Proposition 5.1.2, the value $\sqrt{d_K}L^{\mathrm{alg}}(\psi,0)$ is an algebraic integer and Theorem 5.3.5 becomes

$$2\sqrt{d_K}L^{\mathrm{alg}}(\psi,0)\mathscr{O}_\psi \subset \delta_{\mathscr{O}_\psi}(\mathrm{Eis}_\psi). \tag{5.3.7}$$

In order to compare with the result of Berger we work locally. Let $\mathfrak{p}$ be a split prime in K, dividing a rational prime $p > 3$. Let $K_\mathfrak{p}$ be the completion with ring of integers $\mathscr{O}_\mathfrak{p}$. Let $\mathfrak{q}$ be a prime ideal of K_ψ lying over $\mathfrak{p}$, let $K_{\psi,\mathfrak{q}}$ be the completion with ring of integers $\mathscr{O}_{\psi,\mathfrak{q}}$. Let $v_\mathfrak{p}$ and $v_\mathfrak{q}$ be the valuations in K and K_ψ, related by

$$v_\mathfrak{q}(x) = \frac{1}{[K_{\psi,\mathfrak{q}} : K_\mathfrak{p}]} v_\mathfrak{p}(N_{K_\psi|K}(x)),$$

for $x \in K_\psi$. We can replace $\mathscr{O}_\psi$ by the local ring $\mathscr{O}_{\psi,\mathfrak{p}}$ and define the denominator ideal

$$\delta_{\mathscr{O}_{\psi,\mathfrak{p}}}(\mathrm{Eis}_\psi) \subset \mathscr{O}_{\psi,\mathfrak{p}}.$$

Since p is odd and unramified $2\sqrt{d_K}$ is a $\mathfrak{p}$-adic unit and (5.3.7) becomes

$$L^{\mathrm{alg}}(\psi,0)\mathscr{O}_{\psi,\mathfrak{p}} \subset \delta_{\mathscr{O}_{\psi,\mathfrak{p}}}(\mathrm{Eis}_\psi).$$

In that setting, Berger seems to prove [Ber08, Theorem. 28] the lower bound (in the sense of divisibility)

$$\delta_{\mathscr{O}_{\psi,\mathfrak{p}}}(\mathrm{Eis}_\psi) \subset L^{\mathrm{alg}}(\psi,0)\mathscr{O}_{\psi,\mathfrak{p}},$$

yielding the equality

$$\delta_{\mathscr{O}_{\psi,\mathfrak{p}}}(\mathrm{Eis}_\psi) = L^{\mathrm{alg}}(\psi,0)\mathscr{O}_{\psi,\mathfrak{p}}.$$

Chapter 6

Résumé en français

Cette thèse porte sur l'étude de certains espaces localement symétriques en relation avec des formes automorphes. Une première partie (Chapitres 3 et 4) s'intéresse à des espaces localement symétriques M_K associés à des groupes orthogonaux. Sur ces espaces nous nous intéressons à des formes différentielles qui proviennent de travaux de Mathai et de Quillen et qui permettent de construire certaines séries thêtas à valeur dans la cohomologie de M_K. D'un autre côté, les formes de Mathai et de Quillen permettent aussi de construire des classes d'Eisenstein sur des espaces localements symétriques associés à $\mathrm{SL}_N(K)$ où K est un corps quadratique imaginaire. Dans une seconde partie (Chapitre 5) nous utilisons ces formes pour donner une borne sur le dénominateur de la cohomologie d'Eisenstein.

Partie 1

Espaces localement symétriques associés à des groupes orthogonaux. On se donne un $\mathbb{Q}$-espace quadratique $(X_{\mathbb{Q}}, Q)$ et soit $H := \mathrm{SO}(Q)$ son groupe orthogonal. Soit (p,q) la signature de $X_{\mathbb{R}} = X_{\mathbb{Q}} \otimes \mathbb{R}$. Un sous-espace z de $X_{\mathbb{R}}$ est dit *négatif* si la restriction de Q à z est négative définie. On considère l'espace des sous-espaces vectoriels de dimensions q, négatifs et orientés

$$\mathbb{D} := \left\{ z \subset X_{\mathbb{R}} \quad \text{orienté} \quad \middle| \quad \dim(z) = q, \quad Q\big|_z < 0 \right\}.$$

Cet espace n'est pas connexe et nous noterons $\mathbb{D}^+$ l'une de ses deux composantes connexes. Le groupe $H(\mathbb{R})^+$, la composante connexe du neutre, agit transitivement sur $\mathbb{D}^+$. Si on se donne un point base $z_0 \in \mathbb{D}^+$, le stabilisateur $K_\infty = K_\infty(z_0)$ est un compact maximal de $H(\mathbb{R})^+$ qui est isomorphe à $\mathrm{SO}(p) \times \mathrm{SO}(q)$. On obtient ainsi un isomorphisme

$$\mathbb{D}^+ \simeq H(\mathbb{R})^+/K_\infty \simeq \mathrm{SO}(p,q)^+/\,\mathrm{SO}(p) \times \mathrm{SO}(q).$$

Pour les petites signatures on peut par exemple identifier $\mathbb{D}^+$ avec $\mathbb{R}_{>0}$ pour $(p,q) = (1,1)$, avec $\mathbb{H}$ pour $(p,q) = (2,1)$ ou encore avec $\mathbb{H} \times \mathbb{H}$ pour $(p,q) = (2,2)$. Le troisième exemple est étudié en 4.4.

On se donne ensuite une fonction de Schwartz finie $\varphi_f \in \mathscr{S}(X_{\mathbb{A}_f})$, qui est préservée par un sous-groupe ouvert compact $K_f \subset H(\widehat{\mathbb{Z}})$ agissant par la representation de Weil. Plus concrètement cela signifie que

$$\varphi_f(k^{-1}\mathbf{x}) = \varphi_f(\mathbf{x}) \qquad \forall k \in K_f, \quad \forall \mathbf{x} \in X_{\mathbb{A}_f}.$$

L'espace qui nous intéresse est le double quotient

$$M_K \coloneqq H(\mathbb{Q})\backslash H(\mathbb{A})/K.$$

où $K = K_\infty K_f$. C'est une union disjointe

$$M_K = \bigsqcup_{i=1}^{r} M_{h_i}$$

d'espaces localement symétriques $M_h \coloneqq \Gamma_h \backslash \mathbb{D}^+$ de dimension pq, où Γ_h est l'image dans $H^{\mathrm{ad}}(\mathbb{Q}) = H(\mathbb{Q})/Z(\mathbb{Q})$ de

$$\Gamma_h' \coloneqq H(\mathbb{Q})^+ \cap hK_f h^{-1},$$

et $h_1, \ldots, h_r \in H(\mathbb{A}_f)$ sont tels que

$$H(\mathbb{A}_f) = \bigsqcup_{i=1}^{r} H(\mathbb{Q})^+ h_i K_f. \tag{6.0.1}$$

Le groupe discret $\Gamma_h \subset H(\mathbb{Q})^+$ préserve l'ensemble $\mathcal{L}_h(\varphi_f)$ dans $X_\mathbb{Q}$ défini par

$$\mathcal{L}_h(\varphi_f) \coloneqq \left\{ \mathbf{x} \in X_\mathbb{Q} | \, \varphi_f(h^{-1}\mathbf{x}) \neq 0 \right\}.$$

Notons que si φ_f est la fonction caractéristique d'un réseau adélique, alors $\mathcal{L}_h(\varphi_f)$ est un réseau dans $X_\mathbb{Q}$.

En général M_K est non-compact et on note $\overline{M_K}$ une compactification.

Cycles spéciaux. Pour chaque vecteur $\mathbf{x}$ dans $X_\mathbb{Q}$ avec $Q(\mathbf{x},\mathbf{x}) > 0$ il existe une sous-variété totalement géodésique $\mathbb{D}^+_\mathbf{x} \subset \mathbb{D}^+$ consistant en l'ensemble des $z \in \mathbb{D}^+$ qui sont orthogonaux à $\mathbf{x}$. La codimension de $\mathbb{D}^+_\mathbf{x}$ est q. Ces sous-variétés deviennent des cycles dans M_K lorsque l'on passe au quotient, que nous appelons *cycles spéciaux*. Supposons pour simplifier que $r = 1$ dans (6.0.1).

Ainsi $M_K = \Gamma\backslash\mathbb{D}^+$ est connexe, où $\Gamma' = H(\mathbb{Q})^+ \cap K_f$. Si $\Gamma_{\mathbf{x}}$ est le stabilisateur de $\mathbf{x}$ dans Γ, alors $C_{\mathbf{x}}$ est l'image de la composition

$$\Gamma_{\mathbf{x}}\backslash\mathbb{D}^+_{\mathbf{x}} \hookrightarrow \Gamma_{\mathbf{x}}\backslash\mathbb{D}^+ \longrightarrow \Gamma\backslash\mathbb{D}^+. \tag{6.0.2}$$

Nous noterons aussi

$$C_{\mathbf{x}} \in \mathscr{Z}_{pq-q}(\overline{M_K}, \partial\overline{M_K}; \mathbb{R})$$

le cycle en homologie relative au bord et de codimension q qui est représenté par l'image de l'immersion (6.0.2). Pour $n > 0$ on peut ensuite définir les cycles

$$C_n(\varphi_f) \coloneqq \sum_{\substack{\mathbf{x}\in X_{\mathbb{Q}} \\ Q(\mathbf{x},\mathbf{x})=2n}} \varphi_f(\mathbf{x}) C_{\mathbf{x}} \in \mathscr{Z}_{pq-q}(\overline{M_K}, \partial\overline{M_K}; \mathbb{R}).$$

Dans le cas où M_K a plusieurs composantes connexes, on définit de la même manière un cycle $C_n(\varphi_f, h_i)$ sur M_{h_i} puis on somme sur les composantes connexes de M_K; voir sous-section 2.2.8.

Forme de Kudla-Millson. Dans leurs travaux, Kudla-Millson [KM86; KM87; KM90] construisent une correspondance thêta entre la cohomologie de l'espace M_K et un espace de formes modulaires. Leur construction utilise les travaux de Weil [Wei64] sur les séries thêtas et une forme différentielle φ_{KM}. Plus précisément, la forme différentielle introduite par Kudla et Millson est une q-forme

$$\varphi_{KM} \in \left[\Omega^q(\mathbb{D}^+) \otimes \mathscr{S}(X_{\mathbb{R}})\right]^{H(\mathbb{R})^+}$$

qui est fermée et $H(\mathbb{R})^+$-invariante dans le sens où

$$h^*\varphi_{KM}(\mathbf{x}) = \varphi_{KM}(h^{-1}\mathbf{x})$$

pour tout h dans $H(\mathbb{R})^+$. De plus, c'est une *forme de Thom*, c'est-à-dire que si ω dans $\Omega_c^{pq-q}(\Gamma_{\mathbf{x}}\backslash\mathbb{D}^+)$ est une forme à support compact alors

$$\int_{\Gamma_{\mathbf{x}}\backslash\mathbb{D}^+} \varphi_{KM}(\mathbf{x}) \wedge \omega = 2^{-\frac{q}{2}} e^{-\pi Q(\mathbf{x},\mathbf{x})} \int_{\Gamma_{\mathbf{x}}\backslash\mathbb{D}^+_{\mathbf{x}}} \varphi_{KM}(\mathbf{x}). \tag{6.0.3}$$

Nous parlerons un peu plus des formes de Thom plus tard, lorsque nous expliquerons comment obtenir les formes de Kudla-Millson grâce aux travaux de Mathai et Quillen. Pour le moment rappelons la construction original de Kudla et Millson dans [KM86; KM87].

Soit $\mathfrak{h}$ l'algèbre de Lie de $H(\mathbb{R})^+$ et soit $\mathfrak{k}$ celle de K_∞. Soit $\mathfrak{p}$ une sous-algèbre de Lie telle

$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ soit une décomposition de Cartan, et la somme est orthogonale par rapport à la forme de Killing. L'espace tangent $T_{eK_\infty}\mathbb{D}^+$ de $\mathbb{D}^+$ à l'identité peut être identifié à $\mathfrak{p}$. Ainsi le fibré cotangent est

$$(T\mathbb{D}^+)^* = H(\mathbb{R})^+ \times_{K_\infty} \mathfrak{p}^*,$$

où K_∞ agit sur $\mathfrak{p}^*$ par le dual de la représentation adjointe Ad. On identifie $X_\mathbb{R}$ et $\mathbb{R}^{p+q}$ par une base orthogonale $\mathbf{e}_1, \dots, \mathbf{e}_{p+q}$ telle que les p premiers vecteurs aient norme 1 et les q derniers norme -1. Par rapport à cette base le majorant de Siegel à z_0 est donné par

$$Q^+_{z_0}(\mathbf{x}, \mathbf{x}) \coloneqq \sum_{i=1}^{p+q} x_i^2.$$

Rappelons que $H(\mathbb{R})^+$ agit sur $\mathscr{S}(\mathbb{R}^{p+q})$ par $(h \cdot f)(\mathbf{x}) = f(h^{-1}\mathbf{x})$. On a un isomorphism

$$\left[\Omega^q(\mathbb{D}^+) \otimes \mathscr{S}(\mathbb{R}^{p+q})\right]^{H(\mathbb{R})^+} \longrightarrow \left[\bigwedge^q \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q})\right]^{K_\infty}$$
$$\varphi \longrightarrow \varphi_e$$

en évaluant φ au point base $eK_\infty \in \mathbb{D}^+$, correspondant à z_0. Pour $1 \leq \alpha \leq p$ et $p+1 \leq \mu \leq p+q$ soit $X_{\alpha\mu}$ une base de $\mathfrak{p}$ et $\omega_{\alpha\mu}$ la base duale de $\mathfrak{p}^*$. On définit *l'opérateur de Howe*

$$D\colon \bigwedge^\bullet \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q}) \longrightarrow \bigwedge^{\bullet+q} \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q})$$

par

$$D \coloneqq \frac{1}{2^q} \prod_{\mu=p+1}^{p+q} \sum_{\alpha=1}^{p} A_{\alpha\mu} \otimes \left(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\right)$$

où $A_{\alpha\mu}$ est la multiplication à gauche par $\omega_{\alpha\mu}$, le dual de $X_{\alpha\mu}$. La forme de Kudla-Millson est définie en appliquant D à la gaussienne $\exp\left(-\pi Q^+_{z_0}(\mathbf{x}, \mathbf{x})\right)$:

$$\varphi_{KM}(\mathbf{x})_e \coloneqq D \exp\left(-\pi Q^+_{z_0}(\mathbf{x}, \mathbf{x})\right) \in \bigwedge^q \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q}).$$

Séries thêtas de Kudla-Millson. Voyons maintenant comment la forme de Kudla-Millson permet de construire des séries thêtas à valeur dans la cohomologie de M_K. Nous supposerons que $p+q$ est pair. Rappelons l'existence d'une représentation unitaire

$$\omega\colon \mathrm{SL}_2(\mathbb{R}) \longrightarrow \mathcal{U}(\mathscr{S}(X_\mathbb{R}))$$

appelée la *représentation de Weil*. Elle s'étend en une représentation $[\Omega(\mathbb{D})^+ \otimes \mathscr{S}(X_{\mathbb{R}})]^{H(\mathbb{R})^+}$ et on peut donc considérer la forme

$$\omega(g)\varphi_{KM} \in \left[\Omega(\mathbb{D})^+ \otimes \mathscr{S}(X_{\mathbb{R}})\right]^{H(\mathbb{R})^+}$$

pour $g \in \mathrm{SL}_2(\mathbb{R})$. Pour $h_f \in H(\mathbb{A}_f)$ et $\tau = u + iv$ un point dans le demi-plan posons maintenant la série thêta

$$\Theta_{KM}(\tau, h_f, \varphi_f) \coloneqq v^{-\frac{p+q}{4}} \sum_{\mathbf{x} \in X_{\mathbb{Q}}} \Big(\omega(g_\tau)\varphi_{KM}\Big)(\mathbf{x})\varphi_f(h_f^{-1}\mathbf{x}) \in \Omega(\mathbb{D}^+),$$

où $g_\tau = \begin{pmatrix} \sqrt{v} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{pmatrix}$ est une matrice de $\mathrm{SL}_2(\mathbb{R})$ envoyant i sur τ. Comme $\varphi_{KM}(\mathbf{x})$ est $H(\mathbb{Q})^+$-invariante et $\varphi_f(h_f^{-1}\mathbf{x})$ est $h_f K_f h_f^{-1}$-invariante, il s'en suit que $\Theta_{KM}(\tau, h_f)$ est Γ_{h_f}-invariante, où Γ_{h_f} est l'intersection $H(\mathbb{Q})^+ \cap h_f K_f h_f^{-1}$ comme auparavant. Ainsi, nous pouvons voir $\Theta_{KM}(\tau, h_i)$ comme une forme sur M_{h_i}. En sommant sur les composantes connexes de M_K, on définit ensuite

$$\Theta_{KM}(\tau, \varphi_f) \coloneqq \sum_{i=1}^{r} \Theta_{KM}(\tau, h_i, \varphi_f) \in \Omega^q(M_K). \tag{6.0.4}$$

La forme φ_{KM} étant fermée, il en résulte que $\Theta_{KM}(\tau, \varphi_f)$ est aussi fermée et représente une classe de cohomologie dans $H^q(M_K; \mathbb{R})$. En particulier, si l'on se donne maintenant un cycle (compact) dans $\mathcal{Z}_q(M_K; \mathbb{R})$ on peut considérer la fonction sur le demi-plan

$$\tau \longmapsto \int_C \Theta_{KM}(\tau, \varphi_f) \in \mathbb{C}.$$

Kudla et Millson montrent (dans un cadre plus général) que cette fonction est une forme modulaire de poids $\frac{p+q}{2}$ et d'un certain niveau dépendant de φ_f. On peut réordonner les termes de (6.0.4) pour obtenir le développement en série de Fourier

$$\Theta_{KM}(\tau, \varphi_{KM}) = \sum_{n \in \mathbb{Q}^\times} \Theta_n(v, \varphi_f) e^{2i\pi n\tau},$$

où

$$\Theta_n(v, \varphi_f) \coloneqq \sum_{i=1}^{r} \sum_{\substack{\mathbf{x} \in X_{\mathbb{Q}} \\ Q(\mathbf{x},\mathbf{x}) = 2n}} \varphi_f(h_i^{-1}\sqrt{v}\mathbf{x})\varphi_0(\mathbf{x})$$

et $\varphi^0(\mathbf{x}) \coloneqq e^{\pi Q(\mathbf{x},\mathbf{x})}\varphi_{KM}(\mathbf{x})$. Avec cette normalisation, la propriété de Thom (6.0.3) dit que $\varphi^0(\mathbf{x})$ est un dual de Poincaré de $\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+_{\mathbf{x}}$ dans $\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+$. On peut ensuite vérifier que la forme $\kappa^{-1}\Theta_n(v, \varphi_f)$

est un dual de Poincaré du cycle spécial $C_n(\varphi_f)$ défini plus haut et

$$\kappa := \begin{cases} 2 & \text{si } -1 \in K_f \cap H(\mathbb{Q})^+, \\ 1 & \text{sinon.} \end{cases} \tag{6.0.5}$$

Si l'on intègre cette forme sur C on obtient pour $n > 0$

$$\int_C \Theta_n(v, \varphi_f) = \kappa \langle C_n(\varphi_f), C \rangle$$

et on déduit le développement en série de Fourier de la forme modulaire

$$\int_C \Theta_{KM}(\tau, \varphi_f) = \int_C \Theta_0(v, \varphi_f) + \kappa \sum_{n \in \mathbb{Q}_{>0}} \langle C_n(\varphi_f), C \rangle e^{2i\pi n\tau}. \tag{6.0.6}$$

Les travaux de Kudla-Millson nous permettent ainsi de construire des formes modulaires dont les coefficients de Fourier sont des nombres d'intersection de cycles sur M_K. En revanche, ils sont limités au cas où le cycle C est compact, dans le Chapitre 4 nous nous intéressons à l'integrale le long d'un cycle non-compact.

6.1 La forme de Kudla-Millson et les travaux de Mathai-Quillen

Nous avons vu que la forme $\varphi_{KM} \in [\Omega^q(\mathbb{D}^+) \otimes \mathscr{S}(X_\mathbb{R})]^{H(\mathbb{R})^+}$ est une pièce centrale de la correspondance thêta de Kudla-Millson entre les formes modulaires et la (co)homologie de M_K. Nous avons aussi vu que $\varphi_{KM}(\mathbf{x})$ (ou plutôt la forme $\varphi^0(\mathbf{x})$) était un dual de Poincaré de $\Gamma_\mathbf{x} \backslash \mathbb{D}^+_\mathbf{x}$ dans $\Gamma_\mathbf{x} \backslash \mathbb{D}^+$. Dans les travaus de Kudla et Millson la forme φ_{KM} est construite par la théorie de la représentation et les opérateurs de Howe. Le résultat principal du Chapitre 3 est une construction plus géométrique de la forme de Kudla-Millson grâce à une construction de Mathai et Quillen [MQ86] d'une forme de Thom canonique.

Formes de Thom. Soit $\pi \colon \mathcal{E} \longrightarrow M$ un fibré de rang r sur une variété M connexe lisse et orientée. On suppose que $\mathcal{E}$ est un fibré métrique, c'est à dire qu'il existe une métrique en chaque fibre qui varie de façon lisse, et que $\mathcal{E}$ est orienté. Notons $\Omega_{cv}(\mathcal{E})$ le complexe des formes à support compact dans les fibres (l'indice *cv* signifie '*compact vertical*') et $H_{cv}(\mathcal{E})$ la cohomologie du complexe. En integrant le long de la fibre on obtient un isomorphisme

$$\mathrm{Th} \colon H^{i+r}_{cv}(\mathcal{E}) \longrightarrow H^i(M)$$

appelé *l'isomorphisme de Thom.* Comme $H^0(M) \simeq \mathbb{R}$ on obtient une forme canonique

$$\mathrm{Th}(\mathcal{E}) := \mathrm{Th}^{-1}(1) \in H^r_{cv}(\mathcal{E}),$$

c'est la *classe de Thom* du fibré $\mathcal{E}$. Un représentant de cette classe s'appelle un *forme de Thom.*

On peut remplacer les formes à support vertical compact par des formes rapidement décroissantes dans les fibres, l'essentiel étant que l'intégrale le long des fibres soit convergente. Plus précisément on définit les formes rapidement décroissantes comme suit. Soit $\omega \in \Omega^j(\mathcal{E})$ une forme sur $\mathcal{E}$. Sa retriction $\omega\big|_m$ à une fibre $\mathcal{E}_m := \pi^{-1}(m)$ pour $m \in M$ est une forme sur $\mathcal{E}_m$, qu'on peut donc voir comme un élément de $C^\infty(\mathcal{E}_m) \otimes \wedge^j \mathcal{E}_m$. On dira que ω est rapidement décroissante si ω_m appartient en fait à $\mathscr{S}(\mathcal{E}_m) \otimes \wedge^j \mathcal{E}_m$, et on note $\Omega^j_{rd}(\mathcal{E})$ l'ensemble de ces formes. La fonction

$$\begin{aligned} h \colon \mathcal{E} &\longrightarrow \mathcal{E} \\ \mathbf{y} &\longmapsto \frac{\mathbf{y}}{\sqrt{1 - \|\mathbf{y}\|^2}} \end{aligned}$$

est un difféomorphisme du fibré en disques ouverts $D\mathcal{E}^o$ sur $\mathcal{E}$. Ainsi, par tiré en arrière on obtient un isomorphisme

$$H^i_{rd}(\mathcal{E}) \longrightarrow H^i_{cv}(\mathcal{E}).$$

On peut alors voir la classe de Thom comme un élément dans $H^r_{rd}(\mathcal{E})$. Après avoir fixé une connexion ∇ sur $\mathcal{E}$ qui est compatible avec la métrique, Mathai et Quillen construisent une forme de Thom canonique $U_{MQ} \in \Omega^r_{rd}(\mathcal{E})$.

Un premier résultat. Il y a un fibré naturel sur $\mathbb{D}^+$, le *fibré tautologique* que nous noterons E. Il est défini comme ceci: au dessus de chaque point $z \in \mathbb{D}^+$, la fibre E_z est le plan orienté z. Il s'agit donc d'un fibré de rang q, orienté et qu'on peut munir de la métrique $-Q\big|_z$ en chaque fibre. En appliquant la construction de Mathai-Quillen au fibré E on obtient une forme $U_{MQ} \in \Omega^q_{rd}(E)$ qu'on peut tirer en arrière par une section $s_{\mathbf{x}} \colon \mathbb{D}^+ \longrightarrow E$, qui dépend d'un vecteur $\mathbf{x} \in X_{\mathbb{R}}$. On obtient une forme $s^*_{\mathbf{x}} U_{MQ} \in \Omega^q(\mathbb{D}^+)$, que le théorème suivant relie à la forme de Kudla-Millson.

Théorème A. *(Théorème 3.2.5 dans le texte) On a* $\varphi_{KM}(\mathbf{x}) = 2^{-\frac{q}{2}} e^{-\pi Q(\mathbf{x},\mathbf{x})} s^*_{\mathbf{x}} U_{MQ}$.

En signature $(2, q)$, les espaces sont hermitiens et le résultat fût obtenu par des méthodes similaires dans [Gar18], en utilisant les travaux de Bismut-Gillet-Soulé.

6.2 Restriction à la diagonale de séries d'Eisenstein

Le point de départ du résultat présenté dans le Chapitre 4 est un résultat de Darmon-Pozzi-Vonk qui relie la restriction à la diagonale d'une série d'Eisenstein à des nombres d'intersection de géodésiques. Soit ψ un caractère de Hecke d'order fini et totalement impair sur le groupe de classe restreint $\mathrm{Cl}(F)^+$ d'un corps quadratique réel de discriminant d_F. À un tel caractère on peut associer une série d'Eisenstein $E(\tau_1, \tau_2, \psi)$ qui est une forme modulaire de Hilbert de poids parallèle 1 et de niveau $\mathrm{SL}_2(\mathscr{O})$. Sa restriction à la diagonale $E(\tau, \tau, \psi)$ est une forme modulaire de poids 2 et de niveau $\mathrm{SL}_2(\mathbb{Z})$. Elle est donc nulle, car il n'existe pas d'autres telles formes modulaires. À la place on peut regarder la p-stabilisation $E^{(p)}(\tau_1, \tau_2, \psi)$ à un nombre premier p. La restriction à la diagonale $E^{(p)}(\tau, \tau, \psi)$ est une forme de poids 2 et niveau $\Gamma_0(p)$, non-nulle quand p est scindé. Supposons que p soit scindé et soit $Y_0(p)$ la courbe modulaire $\Gamma_0(p)\backslash\mathbb{H}$. À chaque idéal fractionnaire $\mathfrak{a}$ et à une racine carrée r de D_F modulo p on peut associer une géodésique $\overline{\mathcal{Q}}_{\mathfrak{a},r}$ dans $Y_0(p)$. Soit $\overline{\mathcal{Q}}(\psi)$ le 1-cycle défini par

$$\overline{\mathcal{Q}}(\psi) \coloneqq \sum_{[\mathfrak{a}]\in \mathrm{Cl}(F)^+} \psi(\mathfrak{a})(\overline{\mathcal{Q}}_{\mathfrak{a},r} + \overline{\mathcal{Q}}_{\mathfrak{a},-r}) \in \mathcal{Z}_1(Y_0(p)),$$

et soit $\overline{\mathcal{Q}}(0,\infty)$ l'image dans $Y_0(p)$ de la géodésique qui joint les pointes 0 et ∞. L'identité suivante est prouvée dans [DPV21, Theorem. A]

$$E^{(p)}(\tau, \tau, \psi) = L^{(p)}(\psi, 0) - 2\sum_{n=1}^{\infty} \langle \overline{\mathcal{Q}}(0,\infty), T_n\overline{\mathcal{Q}}(\psi)\rangle e^{2i\pi n\tau} \tag{6.2.1}$$

où $L^{(p)}(\psi, 0) = (1-\psi(\mathfrak{p}))(1-\psi(\mathfrak{p}^\sigma))L(\psi, 0)$ et T_n est un opérateur de Hecke defini par des doubles classes. Dans *loc. cit.* l'égalité (6.2.1) est démontrée en calculant les nombres d'intersections et en les comparant avec les coefficients de Fourier de $E^{(p)}(\tau, \tau, \psi)$. En ayant en tête la construction de Kudla et Millson, on se pose la question suivante.

Question. *Peut-on retrouver l'égalité* (6.2.1) *par le relevé theta de Kudla-Millson?*

L'objectif du Chapitre 4 est de répondre positivement à cette question et de généraliser le résultat de Darmon-Pozzi-Vonk à des corps totalement réels. Pour cela, nous construisons un cycle *relatif* $C \otimes \psi$ qui est un tore dans M_K, tel que l'intégral de la série theta de Kudla et Millson sur ce tore donne la restriction à la diagonale d'une série d'Eisenstein.

Un certain espace quadratique de signature (N, N)**.** À première vue, en comparant les Equations (6.0.6) and (6.2.1), on pourrait penser qu'on doit travailler avec l'espace symétrique $\mathbb{H}$ associé à un groupe orthogonal de signature $(2, 1)$. Mais le bon cadre est en fait de travailler avec l'espace symétrique associé à un groupe orthogonal de signature $(2, 2)$ et d'exploiter l'isomorphisme exceptionel entre $\mathbb{D}^+$ et $\mathbb{H}\times\mathbb{H}$. Pour la généralisation on utilise un espace quadratique de signature (N, N), obtenu par restriction des scalaires comme suit.

Soit $F/\mathbb{Q}$ un espace quadratique totalement réel de degré N, soit $\mathscr{O}$ son anneau des entiers et notons $F_{\mathbb{Q}}$ pour F vu comme $\mathbb{Q}$-algèbre. Soit $X_F^0 = F^2$ l'espace quadratique de dimension 2 sur F, avec la forme quadratique $Q^0(\mathbf{x}, \mathbf{y}) = xy' + x'y$ avec $\mathbf{x} = (x, x')$ et $\mathbf{y} = (y, y')$ des vecteurs dans F^2. À une place v de $\mathbb{Q}$ posons $F_{\mathbb{Q}_v} := F_{\mathbb{Q}} \otimes \mathbb{Q}_v$. L'espace que nous considérons est la restriction des scalaires $X_{\mathbb{Q}} := \mathrm{Res}_{F/\mathbb{Q}} X_F^0 = F_{\mathbb{Q}}^2$ avec la forme quadratique $Q := \mathrm{tr}_{F/\mathbb{Q}} \circ Q^0$. C'est un espace de dimensions $2N$ sur $\mathbb{Q}$ et qui a signature (N, N). On fixe une $\mathbb{Z}$-base $\epsilon_1, \cdots, \epsilon_N$ de $\mathscr{O}$, ce qui identifie $F_{\mathbb{Q}}$ avec $\mathbb{Q}^N$. Cette base identifie $X_{\mathbb{Q}}$ avec $\mathbb{Q}^{2N}$ où la forme quadratique est donnée par la matrice symétrique

$$A(Q) := \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$$

où $A \in \mathrm{GL}_N(\mathbb{Q})$ est une matrice telle que $\det(A) = D_F$ soit le discriminant de F; voir (4.1.3). Comme précédémment nous nous intéressons à l'espace M_K, où K_f préserve une fonction de Schwartz $\varphi_f \in \mathscr{S}(X_{\mathbb{A}_f})$, et à la série thêta $\Theta_{KM}(\tau, \varphi_f) \in \Omega^N(M_K)$.

Un cycle non compact $C \otimes \psi$. Nous intégrons $\Theta_{KM}(\tau, \varphi_f)$ sur un cycle relatif associé à un caractère de Hecke unitaire et d'ordre fini. Pour simplifier ce résumé supposons que ψ est non-ramifié. On suppose aussi que le caractère est totalement impair (autrement dit $\psi_\sigma = \mathrm{sgn}$ à chaque place archimédienne de F), car l'intégrale le long du cycle $C \otimes \psi$ sera nul dès que ψ est pair à une place; voir Remarques 4.3.1 et 4.4.6.

Soit $\mathrm{SO}(F^2) \subset \mathrm{GL}_2(F)$ le groupe orthogonal de l'espace $X^0 = F^2$ avec la forme quadratique Q^0 définie plus haut. L'application

$$\begin{aligned} F^\times &\longrightarrow \mathrm{SO}(F^2) \\ t &\longmapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \end{aligned}$$

est un isomorphisme entre $F^\times$ et $\mathrm{SO}(F^2)$. D'un autre côté le groupe $\mathrm{SO}(F^2)$ peut être naturellement plongé dans $\mathrm{SO}(F_{\mathbb{Q}}^2) \subset \mathrm{GL}_{2N}(\mathbb{Q})$ par restriction des scalaires. En composant les deux plongements et en passant aux adèles on obtient

$$h \colon \mathbb{A}_F^\times \hookrightarrow \mathrm{SO}(F_{\mathbb{A}}^2) \subset \mathrm{GL}_{2N}(\mathbb{A}),$$

où $F^2_{\mathbb{A}} = F^2_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{A}$. Si $h(\widehat{\mathscr{O}}^\times) \subset K_f$, alors le plongement h induit une immersion

$$M_{\mathscr{O}} \longrightarrow M_K, \tag{6.2.2}$$

où

$$M_{\mathscr{O}} \coloneqq F^\times \backslash \mathbb{A}_F^\times / \{\pm 1\}^{N-1} \times \widehat{\mathscr{O}}^\times.$$

L'espace $M_{\mathscr{O}}$ n'est pas connexe et il y a une bijection entre les classes du groupe de classe restreint $\mathrm{Cl}(F)^+$ et les composantes connexes de $M_{\mathscr{O}}$. Plus précisément on peut l'écrire comme une union disjointe

$$M_{\mathscr{O}} = \bigsqcup_{[\mathfrak{a}] \in \mathrm{Cl}(F)^+} \Gamma \backslash \mathbb{R}^N_{>0}$$

où $\Gamma \coloneqq \mathscr{O}^{\times,+}$ sont les unités totalement positives dans $\mathscr{O}$. L'image par l'immersion (6.2.2) de la composante connexe correspondant à $[\mathfrak{a}] \in \mathrm{Cl}(F)^+$ définit un cycle relatif au bord

$$C_{\mathfrak{a}} \in \mathcal{Z}_N(\overline{M_K}, \partial \overline{M_K}; \mathbb{R}).$$

On le tord par le caractère de Hecke pour obtenir

$$C \otimes \psi \coloneqq \sum_{[\mathfrak{a}] \in \mathrm{Cl}(F)^+} \psi(\mathfrak{a}) C_{\mathfrak{a}} \in \mathcal{Z}_N(\overline{M_K}, \partial \overline{M_K}; \mathbb{R}).$$

Les limites des résultats de Kudla-Millson. Nous avons vu que l'intégrale de $\Theta_{KM}(\tau, \varphi_f)$ le long d'un cycle compact est une forme modulaire de poids N dont les coefficients de Fourier sont des nombres d'intersection. Si l'on remplace C par un cycle non-compact, les résultats de Kudla-Millson ne s'appliquent pas directement et trois problèmes peuvent apparaître. Premièrement, l'intégrale peut diverger. Deuxièmement, même si l'intégrale converge la fonction en τ qui en résulte peut être non-holomorphe. Troisièmement, même si l'intégral converge en une fonction holomorphe en τ, il n'est pas clair que les coefficients de Fourier puissent être interprétés en terme de nombre d'intersection de C avec $C_n(\varphi_f)$ comme en (6.0.6). En particulier car un tel nombre n'est à priori pas bien défini, les cycles C et $C_n(\varphi_f)$ étant tous les deux non-compacts.

Dans le cas qui nous intéresse l'intégrale de $\Theta_{KM}(\tau, \varphi_f)$ sur $C \otimes \psi$ ne converge pas, mais peut-être régularisée en ajoutant un paramètre t^s avec $s \in \mathbb{C}$ et en isolant des termes singuliers, comme dans [Kud82]. De plus, bien que les cycles $C_n(\varphi_f)$ et $C \otimes \psi$ soient non-compacts, on montre que le nombre d'intersection $\langle C_n(\varphi_f), C \otimes \psi \rangle$ est bien défini; voir Définition 4.3.14.

Séries d'Eisenstein et restriction à la diagonale. Pour un caractère de Hecke ψ comme en haut et une fonction de Schwartz finie $\phi_f \in \mathscr{S}(F^2_{\mathbb{A},f})$ on définit aussi une série d'Eisenstein

$$E(\tau_1, \ldots, \tau_N, \phi_f, \psi) = E(\tau_1, \ldots, \tau_N, \phi_f, \psi, s)\big|_{s=0}$$

par prolongement analytique, où $(\tau_1, \ldots, \tau_N) \in \mathbb{H}^N$. C'est une forme modulaire de Hilbert de poids parallèle 1. Si l'on prend $\tau = \tau_1 = \cdots = \tau_N$, alors la restriction à la diagonale

$$\tau \longmapsto E(\tau, \ldots, \tau, \phi_f, \psi)$$

est une forme modulaire de poids N.

Comme $X_{\mathbb{A}_f} \simeq F^2_{\mathbb{A},f}$ on peut voir φ_f comme une fonction de Schwartz sur $F^2_{\mathbb{A},f}$. Soient l_1 et l_2 les droites isotropes engendrées par les vecteurs isotropes $\mathbf{e}_1 := {}^{\mathrm{t}}(1,0)$ et $\mathbf{e}_2 := {}^{\mathrm{t}}(0,1)$ dans F^2. Pour une fonction de Schwartz sur $X_{\mathbb{A}_f} \simeq F^2_{\mathbb{A}_f}$ soient φ_1 et φ_2 les fonctions de Schwartz dans $\mathscr{S}(F_{\mathbb{A},f})$ obtenu par restriction de φ_f à l_1 et l_2.

Théorème B. *(Théorème 4.3.12 dans le texte) Soit $\varphi_f \in \mathscr{S}(X_{\mathbb{A}_f})$ telle que φ_1 ou φ_2 soit nulle. L'intégrale*

$$2^{N-1} \int_{C \otimes \psi} \Theta_{KM}(\tau, \varphi_f)$$

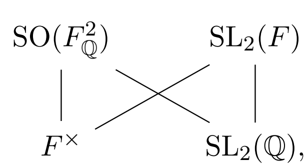
est la restriction à la diagonale de $E(\tau_1, \ldots, \tau_N, \phi_f, \psi)$, où $\phi_f = \mathscr{F}\varphi_f$ est une transformée de Fourier partielle de φ_f. De plus, le développement en série de Fourier de la restriction à la diagonale est

$$E(\tau, \ldots, \tau, \phi_f, \psi) = \zeta_f(\varphi_1, \psi^{-1}, 0) + \zeta_f(\varphi_2, \psi, 0) + (-1)^N 2^{N-1} \kappa \sum_{n \in \mathbb{Q}_{>0}} \langle C_n(\varphi_f), C \otimes \psi \rangle e^{2i\pi n \tau},$$

où ζ_f est une intégrale zeta (voir 4.1.3) et κ est 1 ou 2 comme en (6.0.5).

L'intersection est entre le cycle $C \otimes \psi$ de dimension N et le cycle $C_n(\varphi_f)$ de codimension N (et dimension $N^2 - N$) dans M_K. Le terme constant est constitué de la valeur en $s = 0$ du prolongement analytique de deux fonctions zeta convergentes sur les demis-plans disjoints $\mathrm{Re}(s) > 1$ et $\mathrm{Re}(s) < -1$. La condition d'annulation de φ_1 ou φ_2 garantit que l'un de ces deux termes soit nul et que le prolongement analytique existe bien.

Un seesaw. La série thêta $\Theta_{KM}(\tau, \varphi_f)$ est un noyau thêta pour la paire duale $\mathrm{SL}_2(\mathbb{Q}) \times \mathrm{SO}(F^2_{\mathbb{Q}})$. Le cycle $C \otimes \psi$ est obtenu par le plongement $F^\times \subset \mathrm{SO}(F^2_{\mathbb{Q}})$. On peut résumer ce théorème par le diagramme de *seesaw* suivant



qui relie le noyau thêta $\Theta_{KM}(\tau, \varphi_f)$ à un noyau thêta pour la paire duale $F^\times \times \mathrm{SL}_2(F)$.

Spécialisation aux corps quadratiques. Dans la Section 4.4 on formule le résultat pour un corps quadratique F. On peut alors identifier $F_\mathbb{Q}^2$ avec $\mathrm{Mat}_2(\mathbb{Q})$, l'espace des matrices carrés de taille 2 avec la forme quadratique $2\det$. C'est un espace de signature $(2,2)$ et dans ce cas nous avons déjà mentionné l'isomorphisme entre $\mathbb{D}^+$ et $\mathbb{H} \times \mathbb{H}$. On peut de plus choisir K tel que M_K soit isomorphe au produit de deux courbes modulaires $Y_0(p) \times Y_0(p)$.

Supposons que le premier p soit décomposé. Les deux racines $\pm r$ donnent deux isomorphismes $\mathbb{Z}^2 \simeq \mathscr{O}$, ce qui donne deux cycles $C_{\pm r} \otimes \psi$. Le cycle

$$C_r \otimes \psi + C_{-r} \otimes \psi \in \mathscr{Z}_2\left(\overline{Y_0(p)}^2, \partial\overline{Y_0(p)}^2\right),$$

est alors égal à $\overline{\mathcal{Q}}(\psi) \times \overline{\mathcal{Q}}(\infty, 0)$, où le bord est

$$\partial\overline{Y_0(p)}^2 = \overline{Y_0(p)} \times \partial\overline{Y_0(p)} \cup \partial\overline{Y_0(p)} \times \overline{Y_0(p)}$$

et $\overline{Y_0(p)}$ est la compactification de Borel-Serre de $Y_0(p)$. Pour un bon choix de fonction de Schwartz $\varphi_f^{(p)} \in \mathscr{S}(\mathrm{Mat}_2(\mathbb{A}_f))$, les cycles $C_n(\varphi_f^{(p)})$ sont des correspondances dans $Y_0(p) \times Y_0(p)$ et on a

$$\left\langle C_n(\varphi_f^{(p)}), C \otimes \psi \right\rangle = \left\langle \overline{\mathcal{Q}}(0,\infty), T_n \overline{\mathcal{Q}}(\psi) \right\rangle.$$

De plus, si $\phi^{(p)} \in \mathscr{S}(\mathbb{A}_F^2)$ est la transformée de Fourier partielle de $\varphi^{(p)}$, alors

$$E^{(p)}(\tau, \tau, \psi) = E\left(\tau, \tau, \phi_f^{(p)}, \psi\right)$$

est la p-stabilisation considerée auparavant. Ainsi nous retrouvons le résultat de Darmon-Pozzi-Vonk, voir Corollaire 4.4.12.1

Remark 6.2.1. Dans Corollaire 4.4.12.1 il y a un facteur 4 devant les coefficients de Fourier non-constants, ce qui diffère du facteur 2 dans [DPV21, Theorem. A]. Cela est dû à l'absence du facteur κ dans *loc. cit.*; voir Remarque 4.4.7 pour plus de détails.

Partie 2

Dans le chapitre 5 on considère l'espace localement symétrique Y_Γ associé à $\mathrm{SL}_2(K)$ où $K/\mathbb{Q}$ est un corps quadratique imaginaire. Soit $\mathscr{O}$ son anneau des entiers, que nous supposons ne contenir aucune unité non-triviale, autrement dit que $|\mathscr{O}^\times| = 2$. Soit $Y_\Gamma = \Gamma\backslash\mathbb{H}_3$ où $\Gamma = \mathrm{SL}_2(\mathscr{O})$ et $\mathbb{H}_3$ est l'espace hyperbolique de dimension 3. Soit X_Γ la compactification de Borel-Serre et ∂X_Γ son bord. L'inclusion $Y_\Gamma \longhookrightarrow X_\Gamma$ est une équivalence d'homotopie et donc $H^1(X_\Gamma; \mathbb{C}) \simeq H^1(Y_\Gamma; \mathbb{C})$.

6.3 Borne supérieure sur le dénominateur de la cohomologie d'Eisenstein

Pour toute $\mathscr{O}$ sous-algèbre $R \subset \mathbb{C}$ soit $H(-;R)$ la (co)homologie à coefficients dans R. On a un application de restriction au bord

$$\mathrm{res}\colon H^1(Y_\Gamma;\mathbb{C}) \longrightarrow H^1(\partial X_\Gamma;\mathbb{C}),$$

dont le noyau est la cohomologie intérieure (ou cuspidale) que nous notons $H^1_!(Y_\Gamma;\mathbb{C})$. Elle peut être identifiée avec l'image dans $H^1(Y_\Gamma;\mathbb{C})$ de la cohomologie à support compact $H^1_c(Y_\Gamma;\mathbb{C})$. La cohomologie d'Eisenstein $H^1_{\mathrm{Eis}}(Y_\Gamma;\mathbb{C})$ est un complémentaire de la cohomologie intérieure, tel que l'on ait une somme directe

$$H^1(Y_\Gamma;\mathbb{C}) = H^1_!(Y_\Gamma;\mathbb{C}) \oplus H^1_{\mathrm{Eis}}(Y_\Gamma;\mathbb{C}).$$

Soit ψ un caractère de Hecke algebrique de type z^2 à l'infini et non-ramifié, dont les valeurs sont dans une extension finie F_ψ de K. La cohomolgie d'Eisenstein est définie comme l'image de $\mathrm{Im}(\mathrm{res}) \subset H^1(\partial X_\Gamma;\mathbb{C})$ par l'application Eisenstein de Harder:

$$\mathrm{Eis}_\psi\colon \mathrm{Im}(\mathrm{res}) \longrightarrow H^1(Y_\Gamma;\mathbb{C}).$$

Dans [BCG20; BCG21] Bergeron-Charollois-Garcia utilisent la forme de Mathai-Quillen form pour construire une forme différentielle

$$E_\psi \in \Omega^1(\mathbb{H}^3;\mathbb{C})^\Gamma$$

qui représente une classe d'Eisenstein dans la cohomologie $H^1_{\mathrm{Eis}}(Y_\Gamma;\mathbb{C})$. Plus généralement, ils définissent cette forme sur l'espace symétrique associé à $\mathrm{SL}_N(K)$ et pour des coefficients plus généraux. Dans le cas que nous considérons ($N = 2$ et coeffcients triviaux) cette forme différentielle apparait déjà dans les travaux d'Ito [Ito87]. Elle définit un cocycle $E_\psi \in H^1(\Gamma,\mathbb{C})$ par

$$E_\psi(\gamma) = \int_{u_0}^{\gamma u_0} E_\psi$$

où $u_0 \in \mathbb{H}_3$. Pour un réseau L dans K on définit le cocycle de Sczech $\Phi_L\colon \Gamma \longrightarrow \mathbb{C}$ par

$$\Phi_L\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{cases} I\left(\frac{a+d}{c}\right) E_2(L) - D(a,c,L) & \text{si} \quad c \neq 0, \\ I\left(\frac{b}{d}\right) E_2(L) & \text{si} \quad c = 0, \end{cases}$$

où $E_2(L)$ est une série d'Eisenstein et $D(a, c, L)$ une somme de Dedekind; voir 5.1 pour les définitions. Les deux cocycles sont reliés par

$$E_\psi = \sum_{\mathfrak{a} \in \mathrm{Cl}(K)} \psi(\mathfrak{a})^{-1} \Phi_{\mathfrak{a}}.$$

Cette egalité est démontrée en utilisant l'idée de [BCG21] de déplacer le chemin d'intégration $[u_0, \gamma u_0]$ à l'infini. Plus précisément, on choisit une pointe v de Y_Γ et soit $[v, \gamma^{-1} v]$ le symbole modulaire reliant les pointes v et $\gamma^{-1} v$. Il y a une homotopie entre $[u_0, \gamma u_0]$ et $[v, \gamma^{-1} v]$; voir Figure 5.3. L'intégrale le long du symbole modulaire $[v, \gamma^{-1} v]$ donne la somme de Dedekind, alors que le terme $I\left(\frac{a+d}{c}\right) E_2(\mathscr{O})$ est une contribution des pointes. On retrouve ainsi une formule dûe à Ito [Ito87, Theorem. 3].

On peut utiliser la formule d'Ito pour déduire des résultats d'intégralité sur la cohomologie d'Eisenstein. Sczech a montré que le cocycle $2\Phi_L$ est entier si les séries d'Eisenstein $g_2(L), g_3(L)$ sont des entiers algébriques, voir Proposition 5.3.4. Soit $L^{\mathrm{alg}}(\psi, 0)$ une normalisation de $L(\psi, 0)$ qui soit algébrique. Soit K_ψ une extension de K contenant $L^{\mathrm{alg}}(\psi, 0)$, le corps F_ψ et l'image de Φ_L; soit $\mathscr{O}_\psi$ l'anneau des entiers de K_ψ. Harder montre que Eis_ψ est rationnelle, c'est à dire que la restriction aux coefficients dans K_ψ donne une application

$$\mathrm{Eis}_\psi \colon \mathrm{Im}(\mathrm{res}) \cap H^1(\partial X_\Gamma; K_\psi) \longrightarrow H^1(Y_\Gamma; K_\psi).$$

En se restreignant à des coefficients entiers on a une application

$$\mathrm{Eis}_\psi \colon \mathrm{Im}(\mathrm{res}) \cap \widetilde{H}^1(\partial X_\Gamma; \mathscr{O}_\psi) \longrightarrow H^1(Y_\Gamma; K_\psi), \tag{6.3.1}$$

où $\widetilde{H}(-; \mathscr{O}_\psi)$ est la partie sans torsion de la cohomologie et est identifiée avec

$$\mathrm{Im}\left(H^1(-; \mathscr{O}_\psi) \longrightarrow H^1(-; \mathbb{C})\right).$$

Bien qu'elle soit rationnelle, l'application n'est pas entière, c'est-à-dire que l'image de 6.3.1 n'est pas dans $\widetilde{H}^1(Y_\Gamma; \mathscr{O}_\psi)$. Le dénominateur $\delta_{K_\psi}(\mathrm{Eis}_\psi) \subset \mathscr{O}_\psi$ de la cohomologie d'Eisenstein est l'idéal fractionnaire telle que l'image de (6.3.1) multipliée par le dénominateur est dans $\widetilde{H}^1(X_\Gamma; \mathscr{O}_\psi)$. Si le dénominateur était $\mathscr{O}_\psi$ alors Eis_ψ serait entière, ainsi le dénominateur mesure le défaut d'intégralité.

Théorème C. *(Théorème 5.3.5) Supposons que Cl$(K) = 1$. On a la borne supérieure suivante (dans le sens de divisibilité) sur le dénominateur:*

$$2L^{\mathrm{alg}}(\psi, 0)\mathscr{O}_\psi \subset \delta_{K_\psi}(\mathrm{Eis}_\psi).$$

D'un autre côté, Berger [Ber08] donne une borne inférieure sur le dénominateur, voir Section 5.3.4.

Cela permet d'obtenir une égalité dans certains cas.

Pour le moment on obtient seulement le Théorème 5.3.5 pour nombre de classe 1 et sans coefficients, le cas plus général d'un nombre de classes quelconque et de coefficients plus généraux devrait être accessible par la même méthode. De plus, il devrait en principe être possible de montrer des résultats similaires pour $\mathrm{SL}_N(K)$; voir la remarque ci-dessous.

Remark 6.3.1. Dans leur travaux, Bergeron-Charollois-Garcia [BCG21, Proposition. 3.1] utilisent la classe d'Eisenstein E_ψ sur l'espace associé à $\mathrm{SL}_N(K)$ pour montrer une relation entre une somme de Dedekind et l'intégrale le long d'une géodésique fermée. Ils considèrent une série d'Eisenstein *lissée*, pour la rendre rapidement décroissante à une pointe v. Ils obtiennent ainsi une formule similaire à la formule d'Ito (Théorème 5.2.5) mais sans contributions des pointes. En particulier, en généralisant la démonstration du Théorème 5.2.5 à $\mathrm{SL}_N(K)$ on pourrait obtenir une généralisation de la formule d'Ito et de celle de [BCG21]. Cela pourrait aussi donner des bornes supérieures pour le dénominateur de la cohomologie d'Eisenstein pour $\mathrm{SL}_N(K)$.

Chapter 7

Bibliography

[Ber08] Tobias Berger. "Denominators of Eisenstein cohomology classes for GL2 over imaginary quadratic fields". In: *manuscripta mathematica* 125.4 (2008), pp. 427–470. DOI: 10.1007/s00229-007-0139-6. URL: https://doi.org/10.1007/s00229-007-0139-6.

[Ber09] Tobias Berger. "On the Eisenstein ideal for imaginary quadratic fields". In: *Compositio Mathematica* 145.3 (2009), 603–632. DOI: 10.1112/S0010437X09003984.

[BCG20] Nicolas Bergeron, Pierre Charollois, and Luis E. Garcia. "Transgressions of the Euler class and Eisenstein cohomology of GLN(Z)". In: *Japanese Journal of Mathematics* 15.2 (2020), pp. 311–379. DOI: 10.1007/s11537-019-1822-6. URL: https://doi.org/10.1007/s11537-019-1822-6.

[BCG21] Nicolas Bergeron, Pierre Charollois, and Luis E. Garcia. *Eisenstein cohomology classes for* GL_N *over imaginary quadratic fields*. 2021. arXiv: 2107.01992 [math.NT].

[BGV03] N. Berline, E. Getzler, and M. Vergne. *Heat Kernels and Dirac Operators*. Grundlehren Text Editions. Springer Berlin Heidelberg, 2003. ISBN: 9783540200628.

[BJ06] Armand Borel and Lizhen Ji. *Compactifications of Symmetric and Locally Symmetric Spaces*. Mathematics: Theory and Applications 1. Birkhäuser Boston, MA, 2006.

[BT82] Raoul Bott and Loring W Tu. *Differential forms in algebraic topology*. 1st ed. Vol. 82. Springer-Verlag New York, 1982.

[Bum97] Daniel Bump. *Automorphic Forms and Representations*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997.

[Cas49] JWS Cassels. "A note on the division value of $\wp(u)$". In: *Proc. Cambr. Phil. Soc.* 45 (1949), pp. 167–172.

[Dam71] R. Damerell. "L-functions of elliptic curves with complex multiplication, II". eng. In: *Acta Arithmetica* 19.3 (1971), pp. 311–317. URL: http://eudml.org/doc/205036.

[DPV21] Henri Darmon, Alice Pozzi, and Jan Vonk. "Diagonal restrictions of p-adic Eisenstein families". In: *Mathematische Annalen* 379.1 (2021), pp. 503–548. DOI: 10.1007/s00208-020-02086-2. URL: https://doi.org/10.1007/s00208-020-02086-2.

[Fel05] Dirk Feldhusen. "Nenner der Eisensteinkohomologie der GL_2 über imaginär quadratischen Zahlkörpern". PhD thesis. Rheinischen Friedrich-Wilhelms-Universität Bonn, 2005.

[Fre90] Eberhard Freitag. "Hilbert Modular Forms". In: *Hilbert Modular Forms*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1990, pp. 5–71. ISBN: 978-3-662-02638-0. DOI: 10.1007/978-3-662-02638-0_2. URL: https://doi.org/10.1007/978-3-662-02638-0_2.

[Fun02] Jens Funke. "Heegner Divisors and Nonholomorphic Modular Forms". In: *Compositio Mathematica* 3 (2002), pp. 289–321. DOI: 10.1023/A:1020002121978.

[FM14] Jens Funke and John Millson. "The geometric theta correspondence for Hilbert modular surfaces". In: *Duke Mathematical Journal* 163.1 (2014), pp. 65 –116. DOI: 10.1215/00127094-2405279. URL: https://doi.org/10.1215/00127094-2405279.

[Gar18] Luis E. Garcia. "Superconnections, theta series, and period domains". In: *Advances in Mathematics* 329 (2018), pp. 555 –589. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2017.12.021. URL: http://www.sciencedirect.com/science/article/pii/S0001870816305564.

[Hat01] Allen Hatcher. *Algebraic topology*. Mathematics: Theory and Applications. Cambridge University Press, 2001. URL: https://pi.math.cornell.edu/~hatcher/AT/AT.pdf.

[Ito87] Hiroshi Ito. "A function on the upper half space which is analogous to the imaginary part of $\log \eta(z)$." In: *Journal für die reine und angewandte Mathematik* 373 (1987), pp. 148–165. URL: http://eudml.org/doc/152888.

[Iwa19] Kenkichi Iwasawa. *Hecke's L-functions*. SpringerBriefs in Mathematics. Spring, 1964, Lectures at Princeton University, With a foreword by John Coates and Masato Kurihara. Springer, Singapore, 2019. DOI: 10.1007/978-981-13-9495-9. URL: https://doi.org/10.1007/978-981-13-9495-9.

[JM02] Lizhen Ji and Robert Macpherson. "Geometry of compactifications of locally symmetric spaces". en. In: *Annales de l'Institut Fourier* 52.2 (2002), pp. 457–559. DOI: 10.5802/aif.1893. URL: http://www.numdam.org/articles/10.5802/aif.1893/.

[KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. Reprint of the 1963 original, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996.

[Kud82] S Kudla. "On the integrals of certain singular theta-functions". In: *J. Fac. Sci. Univ. Tokyo* 28 (1982), pp. 439–465.

[Kud84] Stephen S. Kudla. "Seesaw dual reductive pairs". In: *Automorphic forms of several variables*. Vol. 46. Progress in Mathematics. Taniguchi Symposium. Birkhäuser Boston, 1984, pp. 244–268.

[Kud97] Stephen S. Kudla. "Algebraic cycles on Shimura varieties of orthogonal type". In: *Duke Mathematical Journal* 86 (1997), pp. 39 –78. DOI: 10.1215/S0012-7094-97-08602-6. URL: https://doi.org/10.1215/S0012-7094-97-08602-6.

[KM86] Stephen S. Kudla and John J. Millson. "The theta correspondence and harmonic forms. I". In: *Mathematische Annalen* 274.3 (1986), pp. 353–378. DOI: 10.1007/BF01457221. URL: https://doi.org/10.1007/BF01457221.

[KM87] Stephen S. Kudla and John J. Millson. "The theta correspondence and harmonic forms. II". In: *Mathematische Annalen* 277.2 (1987), pp. 267–314. DOI: 10.1007/BF01457364. URL: https://doi.org/10.1007/BF01457364.

[KM90] Stephen S. Kudla and John J. Millson. "Intersection numbers of cycles on locally symmetric spaces and fourier coefficients of holomorphic modular forms in several complex variables". In: *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 71.1 (1990), pp. 121–172. DOI: 10.1007/BF02699880. URL: https://doi.org/10.1007/BF02699880.

[LV80] G. Lion and M. Vergne. *The Weil representation, Maslov index and Theta series*. 1st ed. Progress in Mathematics. Birkhäuser Basel, 1980. ISBN: 978-0-8176-3007-2.

[MQ86] Varghese Mathai and Daniel Quillen. "Superconnections, thom classes, and equivariant differential forms". In: *Topology* 25.1 (1986), pp. 85 –110. ISSN: 0040-9383. DOI: https://doi.org/10.1016/0040-9383(86)90007-8. URL: http://www.sciencedirect.com/science/article/pii/0040938386900078.

[MVW87] C. Moeglin, M.F. Vignéras, and J.L. Waldspurger. *Correspondances de Howe sur un corps p-adique*. Lecture Notes in Mathematics. Springer, 1987. ISBN: 9780387186993.

[Neu99] Jürgen Neukirch. *Algebraic number theory*. Trans. by N. Schappacher. 1st ed. Springer, 1999.

[O'S18] Cormac O'Sullivan. "Formulas for non-holomorphic Eisenstein series and for the Riemann zeta function at odd integers". In: *Research in Number Theory* 4.3 (2018), p. 36.

[PR94] Vladimir Platonov and Andrei Rapinchuk. *Algebraic Groups and Number Theory*. Ed. by Inc Academic Press. Elsevier Science, 1994.

[Pop06] Alexandru A. Popa. "Central values of Rankin L-series over real quadratic fields". In: *Compositio Mathematica* 142.4 (2006), 811–866. DOI: 10.1112/S0010437X06002259.

[Rao93] R. Ranga Rao. "On some explicit formulas in the theory of Weil representation." In: *Pacific Journal of Mathematics* 157.2 (1993), pp. 335 –371.

[Scz84] Robert Sczech. “Dedekindsummen mit elliptischen Funktionen”. In: *Inventiones mathematicae* 76.3 (1984), pp. 523–551. DOI: 10.1007/BF01388472. URL: https://doi.org/10.1007/BF01388472.

[Scz86] Robert Sczech. “Dedekind sums and power residue symbols”. en. In: *Compositio Mathematica* 59.1 (1986), pp. 89–112. URL: http://www.numdam.org/item/CM_1986__59_1_89_0/.

[Ste89] Glenn Stevens. “The Eisenstein measure and real quadratic fields”. In: *Théorie des nombres / Number Theory: Proceedings of the International Number Theory Conference held at Université Laval, July 5-18, 1987*. Ed. by Jean M. de Koninck and Claude Levesque. De Gruyter, 1989, pp. 887–927. DOI: doi:10.1515/9783110852790.887. URL: https://doi.org/10.1515/9783110852790.887.

[Wei64] André Weil. “Sur certains groupes d’opérateurs unitaires”. In: *Acta Mathematica* 111.none (1964), pp. 143 –211.

[Wei76] André Weil. *Elliptic Functions according to Eisenstein and Kronecker*. 1st edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, Berlin, Heidelberg, 1976. ISBN: 978-3-540-07422-9.

[Wei95] André Weil. *Basic Number Theory*. Vol. 144. Grundlehren der mathematischen Wissenschaften 3. Springer Berlin, Heidelberg, 1995.

[Wie85] F. Wielonsky. “Séries d’Eisenstein, intégrales toroïdales et une formule de Hecke”. In: *l’Enseignement Mathématique* 31 (1985), pp. 93–135.

RÉSUMÉ

Nous étudions la cohomologie d'espaces localement symétriques en lien avec des formes automorphes. Dans une première partie, nous nous intéressons à des espaces symétriques associés à des groupes orthogonaux et à la correspondance theta de Kudla et Millson. Elle permet de relever des classes de cohomologie de l'espace localement symétrique à des formes modulaires. Un des ingrédients de cette correspondance theta est une certaine forme différentielle. Un premier résultat nous permet de relier cette forme différentielle à une forme de Thom construite par Mathai et Quillen grace à la théorie de Chern-Weil. Dans un second temps, nous montrons qu'en choisissant une certaine classe de cohomologie associée à un corps de nombre totalement réel F , le relevé par la correspondance theta est la restriction à la diagonale d'une série d'Eisenstein de Hilbert de poids parallèle 1. De plus, nous montrons que ses coefficients de Fourier sont des nombres d'intersections. Lorsque F est un corps quadratique cela nous permet de retrouver un résultat de Darmon, Pozzi et Vonk. Dans une seconde partie, nous nous intéressons à la cohomologie d'Eisenstein d'un espace localement symétrique associé à $\mathrm{SL}_2(K)$, où K est un corps de nombre quadratique imaginaire. Nous donnons dans certains cas une borne inférieure en terme de valeur spéciale de fonction L de Hecke. D'un autre côté, des travaux de Berger donne la même valeur comme borne supérieure, ce qui permet dans certains cas d'obtenir une égalité.

MOTS CLÉS

Espaces localement symétriques, formes modulaires, cohomologie, séries thetas, classes d'Eisenstein, correspondance theta de Kudla-Millson

ABSTRACT

We study the cohomology of locally symmetric spaces in connection with automorphic forms. In a first part, we are interested in symmetric spaces associated to orthogonal groups and in the theta correspondence of Kudla and Millson. It allows us to lift cohomology classes of the locally symmetric space to modular forms. One of the ingredients of this theta correspondence is a certain differential form. A first result allows us to relate this differential form to a Thom form constructed by Mathai and Quillen using Chern-Weil theory. Then, we show that by choosing a certain cohomology class associated to a totally real number field F , its theta lift is the restriction to the diagonal of a Hilbert Eisenstein series of parallel weight 1. Moreover, we show that its Fourier coefficients are intersection numbers. When F is quadratic field this allows us to recover a result of Darmon, Pozzi and Vonk. In a second part, we are interested in the Eisenstein cohomology of a locally symmetric space associated to $\mathrm{SL}_2(K)$, where K is an imaginary quadratic number field. We give in some cases a lower bound in terms of the special value of a Hecke L-function. On the other hand, work by Berger gives the same value as an upper bound, which allows in some cases to obtain an equality.

KEYWORDS

Locally symmetric spaces, modular forms, cohomology, theta series, Eisenstein classes, Kudla-Millson theta lift